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ABOUT SEQUENTIALLY OPEN AND CLOSED SUBSETS IN PRODUCT SPACES

OSMAN MUCUK* AND MINA BETÜL TEKE**
*,** ERCIYES UNIVERSITY, KAYSERI 38030 TURKEY
* ORCID ID: 0000-0001-7411-2871
** ORCID ID: 0009-0000-1168-7686

Abstract. We remind two facts for topological spaces. The one is that in a Hausdorff space X each convergent sequence has a unique limit. This allows us to have a function from the set of all convergent sequences in X to X. Another is that in the first countable spaces, some topological objects such as open subsets, closed subsets, closures and interiors of the sets, continuous functions and many others can be defined in terms of convergent sequences.

In this paper we compare these notions with their sequential versions in topological spaces. We will take the product spaces into account and give some results.

1. Introduction

Convergent sequences are important not only in pure mathematics but also in some others such as information theory, biological science and dynamical systems.

The convergent sequences enable us to give sequential definitions of open and closed subsets; and then to do these for some other topological concepts defined in terms of open and closed subsets. For example continuous maps, connectedness and compactness are among those notions. Sequential definitions of topological objects give us a relief in some proofs and solutions of the problems. Hence many authors have been in afford to find the sequential definitions of some topological objects.

In addition to the convergent sequences, in the literature there exist some varieties of other different types of convergences. The readers are referred for example to a large number of the works [7], Posner [24], Iwinski [17], Srinivasan [25], Antoni [2], Antoni and Salat [3], Spigel and Krupnik [26], Öztürk [27], Savaş and Das [28], Savaş [29], Borsik and Salat [5], [4] [13], Di Maio and Kočinac [19].

Connor and Grosse-Erdmann in [14] replacing the sequential convergence with a function defined on a subspace of the real sequences introduced G-methods. Then following this, Çakallı studied G- continuity in [10] (see also [15] and [11] for some other types of continuities), G-compactness in [12] and the G-connectedness in [9]

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(see also [8]). Mucuk and Şahan in [23] considered the notions of G-open subsets and G-neighbourhoods together with some extra properties of G-continuities.

Lin and Liu in [18] extended G-methods to arbitrary sets rather than topological spaces and presented G-hulls, G-closures, G-kernels and G-interiors. Mucuk and Çakallı recently improved G-connectedness in [21] and G-compactness in [22] for the topological groups with operations which generalises topological groups [6]. The authors in the paper [1] extend these ideas to the direction of neutrosophic topological spaces. We refer [20] and [16] for some sequential definitions and discussions.

In this paper we give an exposition of sequential definitions of some topological notions in product spaces.

We acknowledge that this paper forms some parts of thesis [30].

2. Preliminaries

Let X be a topological space. We use the boldface letters $\mathbf{x}$, $\mathbf{y}$, ... to denote the sequences $\mathbf{x} = (x_n)$, $\mathbf{y} = (y_n)$, ... of the terms in X; and $s(X)$ and $c(X)$ respectively the sets of all sequences and convergent sequences in X. A sequence $\mathbf{x} = (x_n)$ is said to be *convergent* to $\ell \in X$ when any open neighbourhood U of $a \in X$ includes almost all terms of $\mathbf{x}$, that means, except for a finite number of terms, all terms stay in U.

Let A be a subset of X and $x \in X$. The point $x \in X$ is said to be in the *sequentially hull* of A if there exists a sequence $\mathbf{x} = (x_n)$ in A with limit x. The sequentially hull of A is denoted by $[A]^s$ and A is said to be *sequentially closed* if $[A]^s \subseteq A$. Hence A is not sequentially closed whenever there exists a sequence $\mathbf{x} = (x_n)$ in A with a limit ℓ which is not in A.

We note that for $a \in A$, the constant sequence $\mathbf{a} = (a, a, \dots)$ has limit a and therefore we have that $A \subseteq [A]^s$. Hence A is sequentially-closed if and only if $[A]^s = A$. A subset $A \subseteq X$ is called *sequentially open* if $X \setminus A$ is sequentially closed. A subset $U \subseteq X$ is a *sequentially neighborhood* of a if there exists a sequentially open subset A of X such that $a \in A \subseteq U$.

The *sequentially closure* of A, denoted by $\overline{A}^s$, is defined to be the intersection of all sequentially closed subsets containing A, which is also a sequentially closed subset, because the intersection of squentially closed subsets is also sequentially closed. If $A \subseteq K$ and K is a sequentially closed subset, then $[A]^s \subseteq [K]^s \subseteq K$. Taking the intersection of all sequentially closed subsets including A, we conclude that $[A]^s \subseteq \overline{A}^s$.

We remind that a point a in first countable space X is an interior point of the subset A if any sequence $\mathbf{x} = (x_n)$ converging to a is almost in A. Therefore we define a point a in any topological space to be sequential interior point of A and write $a \in A^{0s}$ whenever any sequence $\mathbf{x} = (x_n)$ with limit a is almost in A or equivalently there is no any sequence $\mathbf{x} = (x_n)$ in $X \setminus A$ with limit a.

We say that A is sequentially open if $A \subseteq A^{0s}$. By the fact that the constant sequence $(x_n) = (a, a, \dots)$ converges to a, one can see that $A^{0s} \subseteq A$ and therefore A is sequentially open when $A \subseteq A^{0s}$ or equivalently $A^{0s} = A$.

3. Main Results

Let $X \times Y$ be the product space and $A \times B$ a subset of $X \times Y$. A point (x, y) of $X \times Y$ is said to be in the hull of $A \times B$ if there exists a sequence (a_n, b_n) in $A \times B$ with limit (x, y). The set of all hull points of $A \times B$ is denoted by $[A \times B]^s$. The subset $A \times B$ is *sequentially closed* if $[A \times B]^s \subseteq A \times B$. We can check that for the subsets A and B in $X \times Y$, we have $[A \times B]^s = [A]^s \times [B]^s$

Theorem 3.1. *For a topological space X; and the subsets $A, B \subseteq X$, we have the following*

(i) $[A \cap B]^s \subseteq [A]^s \cap [B]^s$.

(ii) $[A \cup B]^s = [A]^s \cup [B]^s$.

Proof. (i) For an $x \in [A \cap B]_G$, there exists a sequence $\mathbf{x} = (x_n)$ of the terms in $A \cap B$ with the limit x. Hence the sequence $\mathbf{x}$ is in both A and B; and therefore $x \in [A]_G$ and $x \in \cap[B]_G$, which means $x \in [A]^s \cup [B]^s$.

(ii) If $x \in [A \cup B]^s$, then there exists a sequence $\mathbf{x} = (x_n)$ in $A \cup B$ with limit x. Hence we can choose either a subsequence $\mathbf{a} = (a_n)$ in A or a subsequence $\mathbf{b} = (b_n)$ in B with limit x. Otherwise the sequence $\mathbf{x} = (x_n)$ is almost in $X \setminus A$ and $X \setminus B$; and therefore $\mathbf{x} = (x_n)$ is almost in $X \setminus A) \cap X \setminus B = X \setminus (A \cup B)$. This concludes that $x \in [A]^s \cup [B]^s$.

Let $x \in [A]^s \cup [B]^s$. Then either there exists a sequence $\mathbf{a} = (a_n)$ in A or a sequence $\mathbf{b} = (b_n)$ in B with limit x. Hence we can choose a sequence $\mathbf{x} = (x_n)$ in $A \cup B$ with limit x; and therefore $x \in [A \cup B]_G$. □

As a result of this theorem we can say that the finite intersections and unions of sequentially closed subsets are also sequentially closed.

Theorem 3.2. *For a topological space X and subsets $A, B \subseteq X$, we have the following:*

(a) $A \times B \subseteq [A \times B]^s \subseteq \overline{A \times B}^s$;

(b) $A \times B$ is sequentially closed if and only if $[A \times B]^s \subseteq A \times B$;

(c) $A \times B$ is sequentially closed if and only if $[A \times B]^s = A \times B$;

(d) If A and B are closed, then it is $A \times B$ is sequentially closed.

(e) $A \times B$ is sequentially closed if and only each convergence sequence in $A \times B$ has a limit in $A \times B$.

Proof. (a) For any point $(a, b) \in A \times B$, the constant sequence $(a_n, b_n) = ((a, b), (a, b), \dots)$ convergences to (a, b). Hence $(a, b) \in [A \times B]^s$. Further if $(x, y) \in [A \times B]^s$, there exists a sequence (a_n, b_n) in $A \times B$ which converges to (x, y). Hence $(x, y) \in \overline{A \times B}^s$.

(b) This is just the definition of a sequentially closed subset.

(c) This is a direct result of (a) and (b).

(d) If A and B are closed, then $A \times B$ is closed and therefore $\overline{A \times B}^s = A \times B$. Hence by (a) $[A \times B]^s = A \times B$, that means $A \times B$ is sequentially closed.

(e) This is obvious by the definition of a sequentially closed subset. □

Example 3.3. If $X \times Y$ has co-countable topology, then a sequence $(\mathbf{x}, \mathbf{y}) = (x_n, y_n)$ converges to (a, b) if and only if the terms are almost (a, b). Hence all subsets of $X \times Y$ are sequentially closed but not necessarily closed.

Theorem 3.4. Let $X \times Y$ be product topological spaces and let $\{A_i \times B_i \mid i \in I\}$ be a class of sets of $X \times Y$. Then we have the following

(a) $\bigcup_{i \in I} [A_i \times B_i]^s \subseteq [\bigcup_{i \in I} A_i \times B_i]^s$.

(b) $[\bigcap_{i \in I} A_i \times B_i]^s \subseteq \bigcap_{i \in I} [A_i \times B_i]^s$.

Proof. (a) If $(x, y) \in \bigcup_{i \in I} [A_i \times B_i]^s$, then $(x, y) \in [A_{i_0} \times B_{i_0}]^s$ for an $i_0 \in I$ and therefore there is a sequence (a_n, b_n) in $A_{i_0} \times B_{i_0}$ with limit (x, y). That means we have a sequence (a_n, b_n) in $\bigcup_{i \in I} A_i \times B_i$ and therefore $(x, y) \in [\bigcup_{i \in I} A_i \times B_i]^s$.

(b) For $(x,y) \in [\bigcap_{i\in I} A_i \times B_i]^s$, there exists a sequence (a_n, b_n) in $\bigcap_{i\in I} A_i \times B_i$ with limit (x,y). This means (a_n, b_n) is a sequence in each $A_i \times B_i$ for $i \in I$. Hence $(x,y) \in [A_i \times B_i]^s$, and therefore $(x,y) \in \bigcap_{i\in I}[A_i \times B_i]^s$.

$\square$

Theorem 3.5. *For a topological space X and the subsets $A, B \subseteq X$, we have the following*

(i) $(A \cap B)^{0s} \ = \ A^{0s} \cap B^{0s}$.

(ii) $A^{0s} \cup B^{0s} \subseteq (A \cup B)^{0s}$.

Proof. (i) If $a \in (A \cap B)^{0s}$ and $\mathbf{x} = (x_n)$ is a sequence with limit a, then the sequence $\mathbf{x} = (x_n)$ is almost in $A \cap B$. Hence (x_n) is almost in both A and B and therefore $a \in A^0_G \cap B^0_G$.

On the other hand if $a \in A^{0^s} \cap A^{0^s}$ and (x_n) is a sequence with limit a, then (x_n) is almost in both A and B which means (x_n) is almost in $A \cap B$ and therefore $a \in (A \cap B)^{0^s}$

(ii) Let $a \in A^{0s} \cup B^{0s}$ and let the sequence $\mathbf{x} = (x_n)$ have the limit a. $a \in A^{0s}$ means that the sequence $\mathbf{x} = (x_n)$ is almost in A and similarly $a \in B^{0s}$ means that the sequence $\mathbf{x} = (x_n)$ is almost in B. Hence in both case the sequence is almost in $A \cup B$, which means that te sequence $\mathbf{x} = (x_n)$ is almost in A then $\mathbf{x}$ is almost either in A or in B; and therefore $a \in (A \cup B)^{0^s}$

$\square$

As a result of Theorem 3.5 we can state that finite intersections and unions of sequentially open subsets are also sequentially open.

Theorem 3.6. *If X is a topological space and A is a subset $A \subseteq X$, then we have the following:*

(a) $(A \times B)^0 \subseteq (A \times B)^{0^s} \subseteq (A \times B)$*;*

(b) $A \times B$ is sequentially open if and only if $A \times B \subseteq (A \times B)^{0^s}$;

(c) $A \times B$ is sequentially open if and only if $A \times B = (A \times B)^{0^s}$;

(d) If A and B are respectively open in X and Y, then $(A \times B)$ is sequentially open.

Proof. (a) $(A \times B)^0$ is an open subset and therefore if $(a,b) \in (A \times B)^0$, then any sequence converging to (a,b) stays almost in $(A \times B)^0 \subset A \times B$. Hence $(a,b) \in (A \times B)^{0s}$. Moreover if $(a,b) \in (A \times B)^{0s}$, then any sequence converging to (a,b) becomes almost in $A \times B$. Since the constant sequence $(a_n, b_n) = ((a,b),(a,b),\dots)$ has limit (a,b) and therefore $(a,b) \in A \times B$.

(b) This is just the definition of sequentially open subset.

(c) This is a direct result of (a) and (b).

(d) If A and B are open, then $A \times B$ is open in $X \times Y$; and therefore $(A \times B)^0 = A \times B$. Hence by (a), we have that $A \times B = (A \times B)^{0s}$ which means $A \times B$ is sequentially open. $\square$

Example 3.7. Let us consider $X \times Y$ with the co-countable topology. Then any subset $A \times B$ of $X \times Y$ is sequentially open but not necessarily open.

Theorem 3.8. *For a product topological space $X \times Y$, a subset $A \times B$ is sequentially open if and only if $X \times Y \setminus (A \times B)$ is sequentially closed.*

Proof. Assuming $A \times B \subseteq (A \times B)^{0s}$ we need to prove that $[X \times Y \setminus (A \times B)]^s \subseteq X \times Y \setminus (A \times B)$. For $(x,y) \in [X \times Y \setminus (A \times B)]^s$, there exists a sequence (x_n, y_n)

in $X \times Y \setminus A \times B$ with limit (x, y). Hence we have that $(x, y) \in X \times Y \setminus A \times B$. Otherwise if $(x, y) \in A \times B$, then by the assumption $A \times B \subseteq (A \times B)^{0s}$ we have $(x, y) \in (A \times B)^{0s}$ and therefore the sequence (x_n, y_n) is almost in $A \times B$. This is a contradiction since (x_n, y_n) is a sequence in $X \times Y \setminus A \times B$.

On the other hand assume $[X \times Y \setminus A \times B]^s \subseteq X \times Y \setminus A \times B$ and prove that $A \times B \subseteq (A \times B)^{0s}$. If $(a, b) \in A \times B$ and (x_n, y_n) is a sequence with limit (a, b), then the sequence (x_n, y_n) is almost in $A \times B$. Otherwise there exists a subsequence (x_{n_k}, y_{n_k}) of (x_n, y_n) of the terms of $X \times Y \setminus A \times B$ which has limit (a, b) and therefore $(a, b) \in [X \times Y \setminus A \times B]^s \subseteq X \times Y \setminus A \times B$ which means $(a, b) \in X \times Y \setminus A \times B$. This is a contradiction because $(a, b) \in A \times B$. □

Theorem 3.9. Assume that $\{A_i \times B_i \mid i \in I\}$ is a class of the subsets in product space $X \times Y$. Then the following are satisfied.

(a) $(\bigcap_{i \in I} A_i \times B_i)^{0s} \subseteq \bigcap_{i \in I} (A_i \times B_i)^{0s}$.

(b) $\bigcup_{i \in I} (A_i \times B_i)^{0s} \subseteq (\bigcup_{i \in I} A_i \times B_i)^{0s}$.

Proof. (a) Assume that $(a, b) \in (\bigcap_{i \in I} A_i \times B_i)^{0s}$. We prove that $(a, b) \in \bigcap_{i \in I} (A_i \times B_i)^{0s}$. Let (a_n, b_n) be a sequence with limit (a, b). By assumption we have that the sequence (a_n, b_n) is almost in $\bigcap_{i \in I} A_i \times B_i$, and therefore in $A_i \times B_i$ for each $i \in I$. Hence $(a, b) \in (A_i \times B_i)^{0s}$ for each $i \in I$ and therefore $(a, b) \in \bigcap_{i \in I} (A_i \times B_i)^{0s}$.

(b) Assume $(a, b) \in \bigcup_{i \in I} (A_i \times B_i)^{0s}$ and (a_n, b_n) is a sequence with limit (a, b). By assumption $(a, b) \in (A_{i_0} \times B_{i_0})^{0s}$ for an $i_0 \in I$ and therefore the sequence (a_n, b_n) is almost in $A_{i_0} \times B_{i_0}$. That means the sequence (a_n, b_n) is almost in $(\bigcup_{i \in I} A_i \times B_i)$ and therefore $(a, b) \in (\bigcup_{i \in I} A_i \times B_i)^{0s}$. □

4. Conclusion

We call a topological space X *sequentially connected* if it has no any sequentially open and closed proper subset. If X is not connected it has an open and closed proper subset $A \subseteq X$. Hence A is sequentially open and closed; and therefore X is not sequentially connected. Equivalently sequentially connected spaces are connected, but the converse is not always true. For example if X is uncountable set, then with co-countable X is connected but not sequentially connected, because all subsets of X a both re sequentially open and closed.

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Osman Mucuk, Erciyes University
Email address: mucuk@erciyes.edu.tr

Mina BetÜl TEKE, Erciyes Üniversitesity
Email address: mbt5842015@gmail.com