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On the Accuracy and Stability of the Meshless RBF Collocation Method for Neutron Diffusion Calculations

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Abstract

Accuracy and stability are the main properties that make an algorithm preferable to its counterparts in modelling of physical phenomena. The radial basis function collocation method is a novel meshless technique, which exhibits an exponential convergence rate for the numerical solution of partial differential equations. However, it is a global approximation scheme and the ill-conditioning of the collocation matrix may become a serious issue if dense sets of interpolation nodes or high values of shape parameters are utilized. This study discusses four strategies to improve the accuracy and stability of the radial basis function collocation method for the numerical solution of the multigroup neutron diffusion equation. These strategies include using a higher precision value for computations, utilizing higher exponents for the generalized multiquadric, decreasing the value of the shape parameter with the number of nodes and singular value decomposition filtering. The results have shown that by using a higher precision value, choosing a variable shape parameter strategy and filtering the smallest singular values of the collocation matrix it is possible to improve the performance of the meshless collocation method, while increasing the exponent of the multiquadric results in a more accurate but less stable algorithm.

Keywords: Meshless method, RBF collocation, Neutron diffusion, Accuracy, Stability

1. Introduction

The numerical solution of partial differential equations (PDE) has a central role in science and engineering since these equations govern most of the physical phenomena encountered in practical applications. The most widely employed technique to tackle these problems is the finite element method in which the elements that are used to approximate the field variable are connected in a predefined manner. Meshless methods have emerged to eliminate this type of connectivity [1], and they have become a useful and rapidly growing class of numerical methods for the treatment of PDEs.

Meshless methods can be categorized into two groups based on their numerical formulation procedures. Within the weak-form methods, the governing equation is integrated prior to function approximation to minimize the error in an average sense. As a result, the continuity requirement of the trial functions is weakened. On the other hand, for strong-form methods the approximation functions are directly substituted into the PDE. These techniques can also be combined to take advantage of different properties of the algorithms, and the resultant scheme is called a hybrid method. There exist several meshless methods in the literature, and a detailed analysis can be found in [2].

The strong-form radial basis function (RBF) collocation method was proposed by Kansa [3]. Since its introduction, this meshless technique has been used for the numerical solution of various PDEs. When compared with other meshless and conventional methods, the RBF collocation has some significant advantages: 1) Truly meshless: Most of the weak-form methods necessitate the use of a background mesh to perform the numerical integrations. The strong-form nature of the RBF collocation method renders it to be a fully meshless algorithm; 2) Exponential convergence: RBF collocation method exhibits an exponential convergence rate for the solution of PDEs. It is possible to obtain accurate solutions even with a low number of discretization nodes; and 3) Ease of implementation: The method is straightforward so that the programming step is relatively less burdensome.

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The fundamental problem of nuclear engineering is to determine the neutron distribution in a multiplying or nonmultiplying system. This distribution can be obtained by solving the neutron diffusion equation, but the heterogeneous nature of real life problems requires the use of numerical techniques for modelling. The neutron diffusion equation has been solved by both mesh-based [4] and meshless methods [5,6]. In [6] a comparison of the RBF collocation with finite and boundary element methods is presented.

Besides the abovementioned advantageous characteristics of the RBF collocation method, strong-form methods are inherently less stable than weak-form ones, and this stability issue becomes even more important for RBF collocation due to its global approximation property. The effectiveness of a numerical method, whether it is meshless or mesh-based, is evaluated by its accuracy, stability and CPU time usage, and it is necessary to investigate how a numerical algorithm can become more preferable to its opponents. In this study four strategies are discussed to improve the performance of the RBF collocation method for the solution of the neutron diffusion equation. These strategies include the use of i) a higher value of precision, ii) a higher exponent value of the multiquadric basis function, iii) a node number dependent shape parameter strategy and iv) the singular value decompostion (SVD) filtering. The rest of the paper is structured as follows: In section 2, the multigroup neutron diffusion equation is introduced. The radial basis function collocation method and its application to the neutron diffusion equation are presented in section 3. Then, in section 4, discussion of the four strategies is made. The paper is closed up with the conclusions in section 5.

2. Material and Methods

2.1. Multigroup Neutron Diffusion Equation

In 2-D Cartesian geometry, the multigroup neutron diffusion equation with vacuum (Dirichlet) boundary conditions on the right and top sides, and reflective (Neumann) boundary conditions on the left and bottom sides have the following form for a homogeneous medium:

$$\begin{aligned}
&-D_g \nabla^2 \phi_g^{(n)} + \Sigma_{r,g}\phi_g^{(n)} = \sum_{g'=1}^{g-1} \Sigma_{s,g'\to g}\phi_{g'}^{(n)} + s_g, && 0 \le x, y \le a \\
&\frac{\partial \phi_g(x,0)}{\partial y} = 0, && 0 \le x \le a \\
&\phi_g(a,y) = 0, && 0 \le y \le a \\
&\phi_g(x,a) = 0, && 0 < x \le a \\
&\frac{\partial \phi_g(0,y)}{\partial x} = 0, && 0 < y \le a
\end{aligned} \tag{1}$$

where

$$s_g \equiv \begin{cases} \dfrac{1}{k^{(n-1)}}\chi_g F^{(n-1)}, & \text{Multiplying medium} \\ S_{g,ext}, & \text{Nonmultiplying medium} \end{cases} \tag{2}$$

Here $g = 1, \dots, G$ is the energy group index, n is the iteration index, ϕ_g, D_g and $\Sigma_{r,g}$ are the neutron flux, diffusion constant and removal cross section of group g, respectively. $\Sigma_{s,g'\to g}$ is the scattering cross section from group g' to group g, k is the multiplication factor, χ_g is the fission spectrum function of group g, F is the fission source, and $S_{g,ext}$ is an external source term. No upscattering assumption is made, which sets the upper limit of the group to group scattering sum to $g-1$. The fission source is defined by

$$F \equiv \sum_{g'=1}^{G} \nu_{g'} \Sigma_{f,g'} \phi_{g'} \quad (3)$$

where $\nu_{g'}$ and $\Sigma_{f,g'}$ are the number of neutrons emitted per fission and fission cross section of group g', respectively. If the medium is a multiplying one then the neutron diffusion equation is solved iteratively by the fission source iteration method [6]. The iteration is terminated when a predetermined convergence criterion is satisfied. In the contrary case of a nonmultiplying medium, the solution can be obtained directly. When there is only one energy group, the nonmultiplying medium problem simplifies to the Helmholtz equation:

$$-D\nabla^2\phi + \Sigma_a\phi = S_{ext} \quad (4)$$

Here the removal cross section, Σ_r, is reduced to the absorption cross section, Σ_a.

2.2. Radial Basis Function Collocation Method

A function ψ is called radial if it satisfies the following property [7]:

$$\|\boldsymbol{x}_1\| = \|\boldsymbol{x}_2\| \Rightarrow \psi(\boldsymbol{x}_1) = \psi(\boldsymbol{x}_2), \quad \boldsymbol{x}_1, \boldsymbol{x}_2 \in \mathbb{R}^s \quad (5)$$

There are numerous radial functions with different properties proposed for function approximation and solution of differential equations, but the most well-known and widely utilized RBF is the generalized multiquadric (GMQ)

$$\psi_j = (r^2 + c^2)^q \quad (6)$$

where $r = \|\boldsymbol{x} - \boldsymbol{x}_j\|_2$ is the distance between the nodes, c is the shape parameter, q is the exponent, and $\boldsymbol{x}_j$ is the interpolation node. For the special cases of $q = 1/2$ and $q = -1/2$ the function is called multiquadric and inverse multiquadric, respectively. The shape parameter and exponent determine the shape of the GMQ. With an increase in c and q, the function flattens as observed in Figure 1. These parameters also affect the approximation properties of the GMQ. First, for interpolation of functions, it has been shown that as $c \to \infty$ the error introduced by the GMQ vanishes [8]. This ideal property is probable with an infinite precision computation, and in practice, the calculations become less stable as the value of shape parameter increases. Secondly, the exponent q determines the positive definiteness of the GMQ. By Micchelli's theorem [9] it can be shown that the inverse multiquadric is strictly positive definite while the multiquadric is conditionally positive definite of order one.

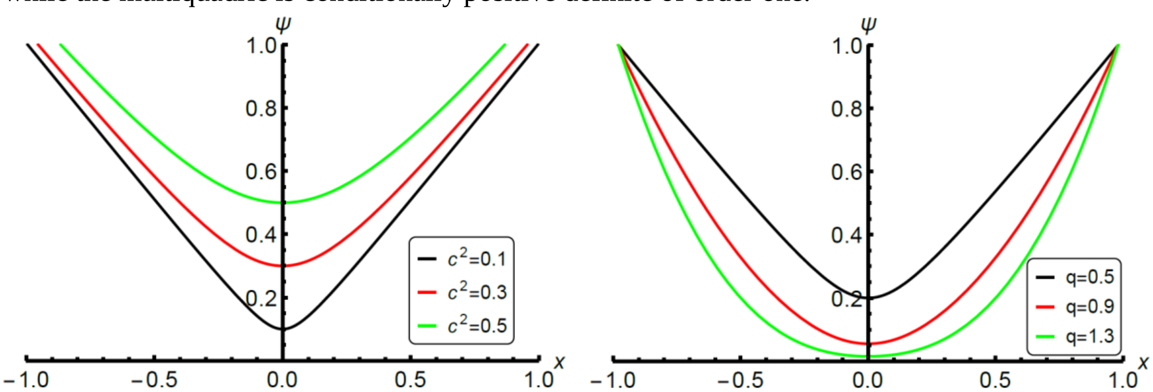

Figure 1. The effect of (left) shape parameter on the multiquadric and (right) exponent on the generalized multiquadric basis function

Now let's consider a PDE and related boundary conditions to introduce the RBF collocation method

$$\begin{aligned} L(u) &= f(\boldsymbol{x}), \quad \boldsymbol{x} \in \Gamma \\ M(u) &= g(\boldsymbol{x}), \quad \boldsymbol{x} \in \Omega \end{aligned} \tag{7}$$

where L and M are differential operators, f and g are known functions, and Γ and Ω represent the domain and boundary of the problem, respectively. The first step of approximation is to create a set of interpolation nodes with N members:

$$\begin{aligned} D &= \{\boldsymbol{x}_1, \dots, \boldsymbol{x}_{N_D}\} \\ B &= \{\boldsymbol{x}_{N_D+1}, \dots, \boldsymbol{x}_{N_D+N_B}\} \\ E &= \{\boldsymbol{x}_{N_D+N_B+1}, \dots, \boldsymbol{x}_N\} \end{aligned} \tag{8}$$

Here N_D and N_B are the numbers of interpolation nodes on the domain and boundary, respectively. As a strong-form method, the RBF collocation is successful in treating Dirichlet type boundary conditions, but the numerical solution may be deteriorated due to the inaccuracies caused by the presence of Neumann type conditions. Fedoseyev et al. [10] have proposed an improved version of the method, which includes PDE collocation on the boundary, and external interpolation nodes to preserve a determined system of equations. From this point of view, a set of external nodes, E, with N_B members are also created to enhance the accuracy of the method.

The next step of the RBF collocation method is the interpolation of the field variable of Eq. (7) with a series of radial basis functions

$$u \cong \sum_{j=1}^{N} a_j \psi_j \tag{9}$$

where a_j are the coefficients to be determined. Substituting Eq. (9) into Eq. (7) and collocating the resultant equations at D and B for the PDE and at B for the boundary conditions yields

$$\begin{aligned} &\sum_{j=1}^{N} a_j L(\psi_{ij}) = f_i, \quad i = 1, \dots, N_D + N_B \\ &\sum_{j=1}^{N} a_j M(\psi_{ij}) = g_i, \quad i = N_D + 1, \dots, N_D + N_B \end{aligned} \tag{10}$$

where $\psi_{ij} = \left(\|\boldsymbol{x}_i - \boldsymbol{x}_j\|_2^2 + c^2\right)^q$, $f_i = f(\boldsymbol{x}_i)$ and $g_i = g(\boldsymbol{x}_i)$. The system of equations, Eq. (10), can be cast into a matrix equation of the form $\boldsymbol{K}\boldsymbol{a} = \boldsymbol{y}$, where $N \times N$ dimensional collocation matrix $\boldsymbol{K}$, $N \times 1$ dimensional coefficient and source vectors $\boldsymbol{a}$ and $\boldsymbol{y}$ are given by

$$\boldsymbol{K} = \begin{bmatrix} L(\psi_{1,1}) & \cdots & L(\psi_{1,N}) \\ \vdots & \vdots & \vdots \\ L(\psi_{N_D+N_B,1}) & \cdots & L(\psi_{N_D+N_B,N}) \\ M(\psi_{N_D+1,1}) & \cdots & M(\psi_{N_D+1,N}) \\ \vdots & \vdots & \vdots \\ M(\psi_{N_D+N_B,1}) & \cdots & M(\psi_{N_D+N_B,N}) \end{bmatrix}, \quad \boldsymbol{a} = \begin{bmatrix} a_1 \\ \vdots \\ a_N \end{bmatrix}, \quad \boldsymbol{y} = \begin{bmatrix} f_1 \\ \vdots \\ f_{N_D+N_B} \\ g_{N_D+1} \\ \vdots \\ g_{N_D+N_B} \end{bmatrix} \tag{11}$$

The solution of $\boldsymbol{K}\boldsymbol{a} = \boldsymbol{y}$ gives the coefficients, $a_i, i = 1, \dots, N$, and hence the numerical solution.

Applying this algorithm to the multigroup neutron diffusion equation and boundary conditions yields

$$\begin{aligned}
&\sum_{j=1}^{N} a_j^{(n)} \left(\Sigma_{r,g} - D_g \nabla^2\right)\psi_{ij} = s_{g,tot,ij} && i = 1, \dots, N_D + N_B \\
&\sum_{j=1}^{N} a_j^{(n)} \frac{\partial \psi_{ij}}{\partial n} = 0 && i = N_D + 1, \dots, N_D + N_R \\
&\sum_{j=1}^{N} a_j^{(n)} \psi_{ij} = 0 && i = N_D + N_R + 1, \dots, N_D + N_B
\end{aligned} \tag{12}$$

where $\partial/\partial n$ is the normal derivative, N_R is the number of nodes on the reflective boundaries and the total source term $s_{g,tot,ij}$ is given by

$$s_{g,tot,ij} = s_g + \sum_{g'=1}^{g-1} \sum_{j=1}^{N} \Sigma_{s,g' \to g} a_j^{(n)} \psi_{ij} \tag{13}$$

Notice that the group-to-group scattering sum is taken into account as a source term due to the no upscattering assumption. The elements of the collocation matrix are thus given by

$$k_{ij} = \begin{cases} \left(\Sigma_{r,g} - D_g \nabla^2\right)\psi_{ij}, & i = 1, \dots, N_D + N_B, \; j = 1, \dots, N \\ \dfrac{\partial \psi_{ij}}{\partial n}, & i = N_D + 1, \dots, N_D + N_R, \; j = 1, \dots, N \\ \psi_{ij}, & i = N_D + N_R + 1, \dots, N_D + N_B, \; j = 1, \dots, N \end{cases} \tag{14}$$

3. Results and Discussion

The four strategies to improve the performance of the RBF collocation solutions of neutron diffusion equation are evaluated by considering a 1-group external source and a 3-group fission source problem. Uniformly scattered sets of nodes with a fill distance of h are utilized and the external nodes are placed at a distance of h to the closest boundary node. The calculations are performed on a domain scaled to $[0,1]^2$ by utilizing the invariance of RBF collocation under uniform scaling. The scaling is done via multiplying $k_{ij}, i = 1, \dots, N_D + N_B, j = 1, \dots, N$ by $1/a^2$. Computations are carried out with MATHEMATICA.

For the external source problem, a constant source of S_0 is chosen. The analytical solution of this case is

$$\phi(x,y) = \left(\frac{4}{\pi}\right)^2 \frac{S_0}{\Sigma_a} \sum_{\substack{l=1 \\ l\ odd}}^{\infty} \sum_{\substack{m=1 \\ m\ odd}}^{\infty} \frac{(-1)^{\frac{l+m}{2}+1}}{lm} \frac{\cos\left(\frac{l\pi x}{a}\right)\cos\left(\frac{m\pi y}{a}\right)}{1 + L^2\left[\left(\frac{l\pi}{a}\right)^2 + \left(\frac{m\pi}{a}\right)^2\right]} \tag{15}$$

where $L = \sqrt{D/\Sigma_a}$ is the diffusion length. For calculations, the upper limits of the infinite series in Eq. (15) are set to 250. The source term, the diffusion length and the diffusion constant are chosen as $1\ n/cm^3 s$, $11.1232\ cm$ and $1.77764\ cm$, respectively. The length of the domain is $a = 25\ cm$. The accuracy of the RBF collocation method is determined via the root mean square (RMS) error:

$$\epsilon_{RMS} = \sqrt{\frac{1}{N_D + N_B} \sum_{i=1}^{N_D + N_B} \left[\phi(x_i, y_i) - \tilde{\phi}(x_i, y_i)\right]^2} \tag{16}$$

where $\tilde{\phi}$ is the numerical neutron flux distribution.

The reference value for the multiplication factor of the 3-group fission source problem can be found by solving

$$\det\begin{bmatrix} D_1B_g+\Sigma_{r,1}-\frac{\chi_1\nu_1\Sigma_{f,1}}{k} & -\frac{\chi_1\nu_2\Sigma_{f,2}}{k} & -\frac{\chi_1\nu_3\Sigma_{f,3}}{k} \\ -\Sigma_{s,1\to2}-\frac{\chi_2\nu_1\Sigma_{f,1}}{k} & D_2B_g+\Sigma_{r,2}-\frac{\chi_2\nu_2\Sigma_{f,2}}{k} & -\frac{\chi_2\nu_3\Sigma_{f,3}}{k} \\ -\Sigma_{s,1\to3}-\frac{\chi_3\nu_1\Sigma_{f,1}}{k} & -\Sigma_{s,2\to3}-\frac{\chi_3\nu_2\Sigma_{f,2}}{k} & D_3B_g+\Sigma_{r,3}-\frac{\chi_3\nu_3\Sigma_{f,3}}{k} \end{bmatrix}=0 \quad (17)$$

In Eq. (17), $B_g = 0.5\times(\pi/a)^2$ is called the geometrical buckling. The nuclear data of this problem is given in Table 1. Diffusion constants and cross sections have units of cm and cm^{-1}, respectively. When $a = 25\ cm$, solution of Eq. (17) yields a multiplication factor of $k = 0.75076$. For calculations, the convergence criterion is set to 10^{-6}.

Table 1. Three-group nuclear data.

Group	D	ν	Σ_f	Σ_r	$\Sigma_{s,g\to g+1}$	$\Sigma_{s,g\to g+2}$	χ
1	3.0034	2.65	0.0131267	0.05286	0.02705	0.01181	0.575
2	2.2297	2.53	0.006102	0.016704	0.00822	-	0.326
3	1.4627	2.47	0.008317	0.01414	-	-	0.099

The error criterion for the fission source problem is the relative absolute percent error in the multiplication factor

$$\epsilon_k = \frac{|k-\tilde{k}|}{k}\times 100 \quad (18)$$

where $\tilde{k}$ is the numerical multiplication factor.

3.1 The Effect of Precision on the Stability of the Method

A brute force method to improve the stability of the solution is to increase the precision. Generally, the matrix condition number is used to test whether a numerical method is stable or not. For an algebraic system of equations, the relative round-off error can be estimated by [11]

$$\epsilon_{round} \leq \kappa\epsilon_{mach} \quad (19)$$

where κ is the condition number and ϵ_{mach} is the machine precision. Using a higher precision arithmetic decreases the machine precision and therefore it provides a more stable computation environment, since for a specified round-off error it is possible to achieve higher condition numbers.

In order to test the effect of precision on the stability of the method, the external source problem with the constant source is considered. Figure 2 shows the comparison of results obtained by FORTRAN's double precision and MATHEMATICA's 100-precision arithmetic. In these calculations the shape parameter of the multiquadric radial basis function is chosen as $c^2 = 0.1$. It is clear from this figure that using a higher precision has improved the stability of the method. For the double precision, the RMS error has started to increase above $h^{-1} = 25$ whereas it has continued to decrease when calculations are done with 100-precision.

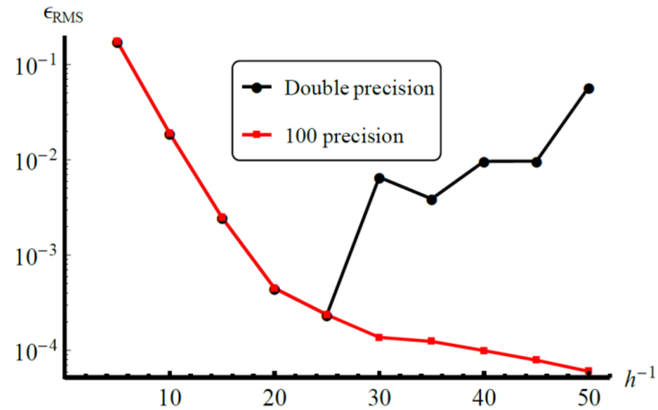

Figure 2. Comparison of double and 100-precision calculations in the RBF collocation method

The price to pay when the precision is increased is the CPU time, as expected. To see the relation between precision and CPU time, several numerical tests are performed with different precision values, and it is found that higher precision arithmetic becomes worse in terms of CPU time as the number of nodes used in discretization gets higher. As an example, for $h^{-1} = 50$, the CPU time is 20% more when 100- and 20-precision computations are compared.

3.2 Exponent of the Generalized Multiquadric

The shape parameter, c, has a vital role in the RBF collocation method, and many studies focus on the effects of this parameter in order to improve the accuracy and stability of the numerical solutions [12-14]. Although the exponent of the generalized multiquadric basis function has a geometrical effect similar to that of the shape parameter, most of the works utilize the multiquadric and inverse multiquadric basis functions without considering the possible effects of the exponent, q.

To see the influence of q on the accuracy and stability of the algorithm the 3-group fission source problem is dealt with. The shape parameter of the GMQ is chosen as $c^2 = 0.03$. In Figure 3, the variation of relative absolute percent error with respect to h^{-1} is plotted for three values of the exponent. It is observed from this figure that the accuracy of the method increases with increasing q value, however the algorithm becomes less stable. When $q = 0.5$ (i.e., the MQ case), the RMS error decreases with the fill distance continuously, while it started to increase above $h^{-1} = 40$ and $h^{-1} = 35$, for $q = 1.5$ and $q = 2.5$, respectively. It can also be deduced from Figure 3 that increasing the exponent enhances the convergence rate of the collocation method. These results show that it is possible to improve the accuracy of the RBF collocation method in expense of stability by increasing the exponent of the generalized multiquadric radial basis function.

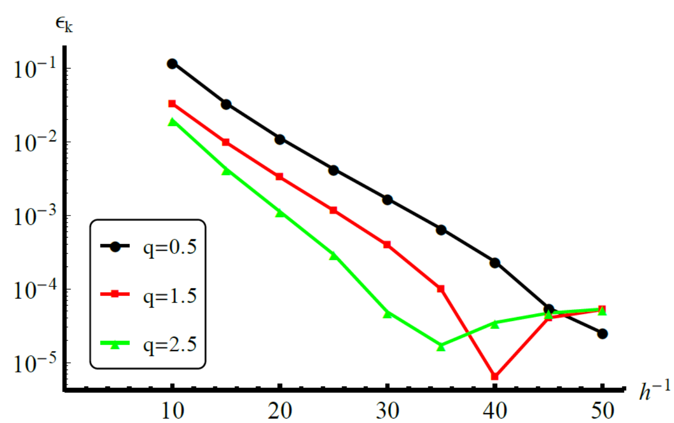

Figure 3. The variation of ϵ_k with respect to h^{-1} for the 3-group fission source problem

3.3 Node Number Dependent Shape Parameter Strategy

Radial basis functions can be expressed in different forms. For instance, the GMQ can be stated as

$$\psi_j = \left[r^2\left(\frac{\alpha}{h}\right)^2 + 1\right]^q \tag{20}$$

where α is called the relative width parameter since it is the width relative to the fill distance [15]. It is obvious that $\alpha = ah$ corresponds to a constant shape parameter approach where a is an arbitrary constant. In [15], six strategies were tested to treat the Runge phenomenon which is a source of accuracy degradation for numerical methods. One of these strategies is to use a variable α scheme instead of a constant one. It was found that by decreasing α as $a/N^{1/4}$, the Runge phenomenon can be defeated in interpolation of functions.

This node number dependent shape parameter approach is examined for the solution of the 3-group fission source problem. The multiquadric is used as the basis function. The ϵ_k values of this strategy are presented in Figure 4 together with those of constant α and constant shape parameter (i.e., $\alpha = h$) strategies. This figure shows that the variable $\alpha = 1/N^{1/4}$ scheme provides much more accurate and stable results than those of constant α approach when $h^{-1} > 10$. In addition to this, although the $\alpha = h$ approach yields higher accuracy, the error starts to oscillate above $h^{-1} = 20$. Thus, when encountered with an instability due to constant shape parameters, it is possible to stabilize the numerical algorithm by decreasing the shape parameter with N.

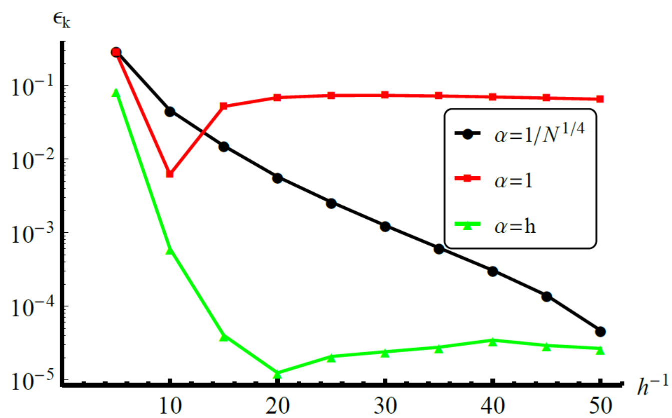

Figure 4. ϵ_k values of the node number dependent shape parameter, $\alpha = 1$ and $\alpha = h$ approaches for the 3-group fission source problem

3.4 Singular Value Decomposition Filtering

For an $m \times n$ matrix $\boldsymbol{A}$ the SVD is defined by

$$\boldsymbol{A} = \boldsymbol{U}^T \boldsymbol{\Sigma} \boldsymbol{V} \tag{21}$$

where $\boldsymbol{U}$ and $\boldsymbol{V}$ are orthogonal matrices and $\boldsymbol{\Sigma}$ is a square diagonal matrix containing the singular values. These matrices satisfy [16]

$$\boldsymbol{U}^T \boldsymbol{A} \boldsymbol{V} = \boldsymbol{\Sigma} = \mathrm{diag}(\sigma_1, \dots, \sigma_p) \in \mathbb{R}^{m \times n}, \quad p = \min\{m, n\} \tag{22}$$

where $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_p \geq 0$. In [17] it has been shown that SVD can be used as a tool for teaching linear algebra geometrically, and also it is applied in solving least squares problems and data compression.

SVD is an effective tool for solving linear systems when the matrix in question is ill-conditioned. Since the RBF collocation method is a global approximation scheme, it gives a full matrix at the end of discretization process. The solution can become unstable depending on the values of the fill distance, h and shape parameter, c. Hence, SVD may improve the performance of the algorithm by treating the ill-conditioning of the collocation matrix.

Now suppose that the linear system resulting from approximation of a PDE with its BCs is given by

$$\boldsymbol{K}\boldsymbol{a} = \boldsymbol{f} \tag{23}$$

If this system is decomposed into its singular values one has

$$\boldsymbol{U}^T \boldsymbol{\Sigma} \boldsymbol{V} \boldsymbol{a} = \boldsymbol{f} \tag{24}$$

and the vector whose elements are the coefficients of the RBFs can be found by

$$\boldsymbol{a} = \boldsymbol{V}^T \boldsymbol{\Sigma}^{-1} \boldsymbol{U} \boldsymbol{f} \tag{25}$$

When the condition number of $\boldsymbol{K}$ is high, it is useful to omit the smallest singular values by replacing $1/\sigma_j$ with zero in $\boldsymbol{\Sigma}^{-1}$**.** By performing this SVD filtering, the amplification of round-off errors corresponding to the smallest singular values is depressed [15].

Numerical experiments are performed with the multiquadric to see the effect of SVD filtering with 50-precision arithmetic for the constant external source case. The fill distance is chosen to be $h^{-1} = 15$ which means that there are 320 singular values. For these fill distance and precision values instability is observed when $c^2 \geq 0.8$ if all singular values are kept. The contour plot in Figure 5 demonstrates the RMS error with respect to the shape parameter, c, and the number of singular values omitted in calculations, n_{sv}. This figure shows that SVD filtering can improve the accuracy of the RBF collocation method in both stable and unstable regions. The results also show that as the shape parameter increases (i.e., the collocation matrix becomes more ill-conditioned), n_{sv} has to be increased to obtain the best results.

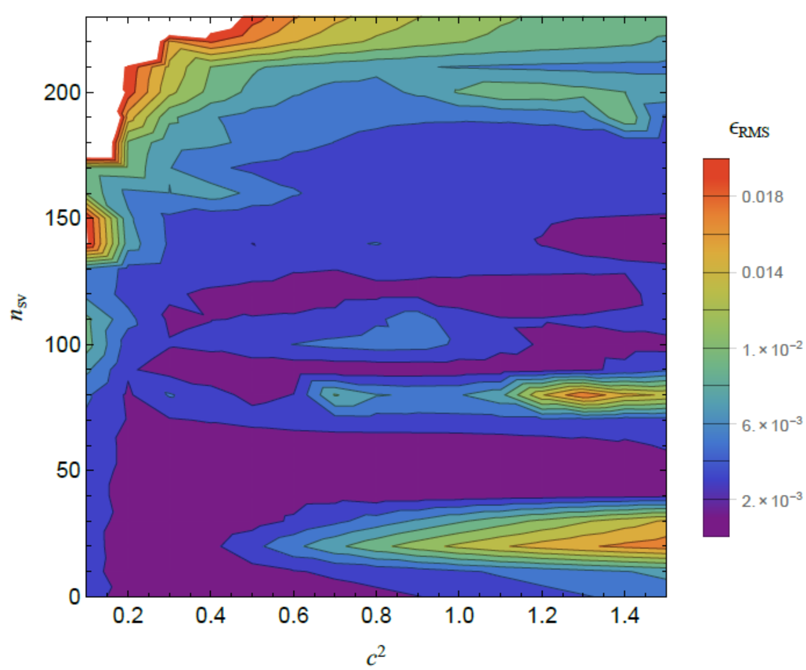

Figure 5. RMS error of constant source problem with respect to c and n_{sv}

4. Conclusions

In this paper, four strategies are considered to improve the accuracy and stability of the meshless radial basis function collocation method for the numerical solution of the multigroup neutron diffusion equation. Both external and fission source problems are studied. The results of the numerical experiments can be summarized as follows:

- Increasing the precision of the calculations has resulted in a more stable solution for the sake of CPU time.
- The exponent of the generalized multiquadric has a similar effect to that of the shape parameter since a tradeoff is observed between the accuracy and the stability of the method with an increasing exponent value.
- The collocation method can be stabilized by utilizing a node number dependent shape parameter strategy in which the value of the shape parameter decreases as a denser set of interpolation nodes is employed.
- When the collocation matrix becomes ill-conditioned, the accuracy of the algorithm can be enhanced by the singular value decomposition filtering technique.

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