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THE EXTENDED ODD WEIBULL-G FAMILY: PROPERTIES AND APPLICATIONS

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Abstract. The Weibull distribution is one of the most popular and widely used model for failure time in life-testing and reliability theory. In this study, we introduce a new class of continuous distributions called the *extended odd Weibull-G* family. Special models of new family are provided. Various structural properties including explicit expressions for the ordinary and incomplete moments, generating function, Rényi and Shannon entropies, order statistics and probability weighted moments are derived. The maximum likelihood method is used for estimating model parameters. The flexibility of the generated family is illustrated by means of two applications to real data sets.

1. Introduction

In many applied sciences such as medicine, engineering and finance, among others, modeling and analyzing lifetime data are crucial. Many generalized families of distributions have been proposed and studied over the last two decades for modeling lifetime data in many applied areas such as economics, engineering, biological studies, environmental sciences, medical sciences and finance. Some well-known families are the Marshall-Olkin-G by Marshall and Olkin (1997), the beta-G by Eugene *et al.* (2002), the transmuted-G by Shaw and Buckley (2007), the gamma-G by Zografos and Balakrishnan (2009), the Kumaraswamy-G by Cordeiro and de Castro (2011), the McDonald-G by Alexander *et al.* (2012), the Weibull-G by Bourguignon *et al.* (2014), the exponentiated half-logistic generated family by Cordeiro *et al.* (2014), the beta odd log-logistic generalized by Cordeiro *et al.* (2015) and the generalized odd log-logistic-G by Cordeiro *et al.* (2017). Several mathematical properties of the extended distributions may be easily explored using mixture forms of exponentiated-G (Exp-G) distributions. However, there still remain many important problems involving real data, which do not follow any of these families. In

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fact, there are not many flexible distributions to model left-skewed data in statistics literature.

The exponential distribution does not provide enough flexibility for analyzing different types of lifetime data because of having constant hazard rate. So, in the literature there are many generalizations that makes the exponential distribution more flexible. For example, Cordeiro *et al.* (2012) introduced a new extension of the exponential distribution which is called as the extended exponential (EE) distribution. The cumulative distribution function (cdf) and probability density function (pdf) of the EE distribution are, respectively, given by

$$F_{EE}(x) = 1 - (1 + \beta\,\lambda\,x)^{-1/\beta}$$

$$f_{EE}(x) = \lambda\,(1 + \beta\,\lambda\,x)^{-1-1/\beta}$$

where $x > 0$ and $\beta, \lambda > 0$. When $\beta \to 0^+$, it reduces to exponential distribution.

Now, motivated by Cordeiro *et al.* (2012), we define Extended Weibull (ExW) distribution. The cdf and pdf of the ExW distribution are, respectively, given by

$$F_{ExW}(x) = 1 - (1 + \beta\,\lambda\,x^{\alpha})^{-1/\beta}$$

$$f_{ExW}(x) = \alpha\lambda\,x^{\alpha-1}(1 + \beta\,\lambda\,x^{\alpha})^{-1-1/\beta}$$

where $\lambda > 0$ is a scale parameter, $\alpha, \beta > 0$ are two shape parameters and $x > 0$. When $\beta \to 0^+$, it reduces to Weibull distribution.

The aim of the paper is to propose a new flexible family of distributions using the ExW distribution. In this way, we will utilize the flexibility of the baseline distribution for modelling the data. The new family is called as the *extended odd Weibull-G* (ExOW-G for short) family and a comprehensive description of its mathematical properties is given. In fact, the new ExOW-G family is motivated by its ability to model data with with increasing, decreasing, unimodal, bimodal shaped failure rates. Furthermore, the special models of this family are shown to provide better fits than other competitive models generated by other well-known families in the literature. The new family due to its flexibility in accommodating different forms of the risk function seems to be an important family that can be used in a variety of problems in modeling survival data.

Let $G\,(x;\boldsymbol{\xi})$ and $g\,(x;\boldsymbol{\xi})$ are the baseline cdf and pdf belong a random variable, respectively. Based on the (T-X) generator of Alzaatreh *et al.* (2013) and using the ExW distribution when λ=1, the cdf and pdf of the ExOW-G family are given (for $x > 0$) by

$$F(x;\alpha,\beta,\boldsymbol{\xi}) = \int_0^{\frac{G(x;\boldsymbol{\xi})}{\bar{G}(x;\boldsymbol{\xi})}} \alpha\,t^{\alpha-1}(1+\beta\,t^{\alpha})^{-1-1/\beta}dt = 1 - \left\{1 + \beta\,\left[\frac{G(x;\boldsymbol{\xi})}{\bar{G}(x;\boldsymbol{\xi})}\right]^{\alpha}\right\}^{\frac{-1}{\beta}} \tag{1.1}$$

and

$$f(x;\alpha,\beta,\boldsymbol{\xi}) = \frac{\alpha\,g(x;\boldsymbol{\xi})\,G(x;\boldsymbol{\xi})^{\alpha-1}}{\bar{G}(x;\boldsymbol{\xi})^{\alpha+1}}\,\left\{1 + \beta\left[\frac{G(x;\boldsymbol{\xi})}{\bar{G}(x;\boldsymbol{\xi})}\right]^{\alpha}\right\}^{\frac{-1}{\beta}-1}, \tag{1.2}$$

respectively, where $\bar{G}\,(x;\boldsymbol{\xi}) = 1 - G\,(x;\boldsymbol{\xi}),\ g(x;\boldsymbol{\xi}) = dG(x;\boldsymbol{\xi})/dx,\ \alpha,\beta > 0$ and $\boldsymbol{\xi}$ denote the vector of parameters for baseline cdf G. Henceforth, a random variable with density (1.2) is denoted by $X \sim$ExOW-G$(\alpha,\beta,\boldsymbol{\xi})$. When $\beta \to 0^+$ we obtain the Weibull-G (W-G) family of distributions.

The reliability function (rf), hazard rate function (hrf) and cumulative hazard rate function (chrf) of X are, respectively, given by

$$R(x;\alpha,\beta,\boldsymbol{\xi}) = \left\{1+\beta\left[\frac{G(x;\boldsymbol{\xi})}{\bar{G}(x;\boldsymbol{\xi})}\right]^{\alpha}\right\}^{\frac{-1}{\beta}},$$

$$h(x;\alpha,\beta,\boldsymbol{\xi}) = \frac{\alpha\, g(x;\boldsymbol{\xi})\, G(x;\boldsymbol{\xi})^{\alpha-1}}{\bar{G}(x;\boldsymbol{\xi})^{\alpha+1}\left\{1+\beta\left[\frac{G(x;\boldsymbol{\xi})}{\bar{G}(x;\boldsymbol{\xi})}\right]^{\alpha}\right\}}, \tag{1.3}$$

and

$$H(x;\alpha,\beta,\boldsymbol{\xi}) = \frac{1}{\beta}\log\left\{1+\beta\left[\frac{G(x;\boldsymbol{\xi})}{\bar{G}(x;\boldsymbol{\xi})}\right]^{\beta}\right\}.$$

An interpretation of the ExOW-G family can be given as follows: Let T be a random variable describing a stochastic system by the cdf $G(x)$. If the random variable X represents the odds ratio, the risk that the system following the lifetime T will be not working at time x is given by $G(x)/[1-G(x)]$. If we are interested in modeling the randomness of the odds ratio by the density function of the ExW distribution $r(t) = \alpha\, t^{\alpha-1}(1+\beta\, t^{\alpha})^{-1-1/\beta}$ (for $t>0$), the cdf of X is given by

$$Pr(X \le x) = R\left(\frac{G(x)}{1-G(x)}\right),$$

which is exactly the cdf of the new family.
"Furthemore, the basic motivations for using the ExOW-G family in practice are the following:

- ✓ to make the kurtosis more flexible compared to the baseline model;
- ✓ to produce a skewness for symmetrical distributions;
- ✓ to construct heavy-tailed distributions that are not longer-tailed for modeling real data;
- ✓ to generate distributions with symmetric, left-skewed, right-skewed and reversed-J shaped;
- ✓ to define special models with all types of the hrf;
- ✓ to provide consistently better fits than other generated models under the same baseline distribution."

Theorem 1 provides some relations of the ExOW-G family with other distributions.

Theorem 1. *Let* $X \sim$*ExOW*-$G(\alpha, \beta, \boldsymbol{\xi})$.

(a) If $Y = \left[\frac{G(X;\boldsymbol{\xi})}{G(X;\boldsymbol{\xi})}\right]^{\alpha}$, *then* $F_Y(y) = 1 - (1 + \beta\, y)^{-1/\beta} \quad y > 0$;

(b) If $Y = \dfrac{G(X;\boldsymbol{\xi})}{G(X;\boldsymbol{\xi})}$, *then* $Y \sim ExW$.

The rest of the paper is organized as follows. In Section 2, we give useful linear representation for the family density function. In Section 3, we present two special models and plots of their pdfs and hrfs. In Section 4, we derive some of its general mathematical properties including asymptotics, ordinary and incomplete moments, quantile and generating functions, quantile power series, entropies, order statistics and probability weighted moments (PWMs). Maximum likelihood estimation (MLE) of the model parameters is addressed in Section 5. Simulation results to assess the performance of the maximum likelihood method are reported in Section 6. In Section 7, we provide two applications with real data sets to illustrate the flexibility of the new family. Finally, we give some concluding remarks in Section 8.

2. Linear representation

In this section, we provide a useful representation for the pdf of ExOW-G family. Consider the series expansion, we can write

$$Z^q = \sum_{k=0}^{\infty} \frac{(q)_k}{k!} (z-1)^k, \tag{2.1}$$

where $(q)_k = q(q-1)\cdots(q-k+1)$.

Applying the power series (2.1) to (1.1), we get

$$F(x) = 1 - \sum_{k=0}^{\infty} \frac{(-1/\beta)_k}{k!} \beta^k G(x)^{\alpha k} \, [1 - G(x)]^{-\alpha k}. \tag{2.2}$$

Consider the power series

$$(1-z)^{-q} = \sum_{n=0}^{\infty} \frac{\Gamma(q+n)}{n!\,\Gamma(q)} z^n. \tag{2.3}$$

After applying the power series (2.3) to $[1 - G(x)]^{-\alpha k}$, we have

$$[1 - G(x)]^{-\alpha k} = \sum_{j=0}^{\infty} \frac{\Gamma(\alpha k + j)}{j!\Gamma(\alpha k)} G(x)^j. \tag{2.4}$$

Substituting (2.4) in (2.2), we get

$$F(x) = 1 - \sum_{k,j=0}^{\infty} \frac{\beta^k \Gamma\left(\alpha k + j\right)\left(-1/\beta\right)_k}{k!j!\Gamma\left(\alpha k\right)} G(x)^{\alpha k + j} .$$

Then, the pdf of ExOW-G family can be expressed as a mixture of Exp-G family as

$$F(x) = 1 - \sum_{k,j=0}^{\infty} b_{k,j}\ H_{\alpha k + j}(x), \tag{2.5}$$

where $b_{k,j} = \beta^k \Gamma\left(\alpha k + j\right)\left(-1/\beta\right)_k / k!j!\Gamma\left(\alpha k\right)$ and $H_{\alpha k+j}\left(x\right)$ is the cdf of the Exp-G family with power parameter $\alpha k + j$. By differentiating (mixture), the pdf in (1.2) can be expressed as

$$f(x) = \sum_{k,j=0}^{\infty} a_{k,j}\ h_{\alpha k + j}(x),$$

where $a_{k,j} = -b_{k,j}$ and $h_{\alpha k+j}\left(x\right) = \left(\alpha k + j\right) g(x) G(x)^{\alpha k + j - 1}$ is the Exp-G density with power parameter $\alpha k + j$. Thus, several mathematical properties of the ExOW-G family can be obtained simply from those properties of the Exp-G family.

3. Special models

In this section, we provide two special models of the ExOW-G family. The pdf in (1.2) will be most tractable when the cdf $G(x)$ and the pdf $g(x)$ have simple analytic expressions.

3.1. **The ExOW-Normal (ExOW-N) distribution.** The ExOW-N distribution is defined from (1.2) by taking $G(x;\mu,\sigma^2) = \Phi\left(\frac{x-\mu}{\sigma}\right)$ and $g(x;\mu,\sigma^2) = \sigma^{-1}\phi\left(\frac{x-\mu}{\sigma}\right)$ for the cdf and pdf of the normal (N) distribution with a location parameter $\mu \in \mathbb{R}$ and a scale positive parameter σ^2, where $\Phi\left(.\right)$ and $\phi\left(.\right)$ are the cdf and pdf of the standard N distribution, respectively. The pdf of ExOW-N distribution is given (for $x \in \mathbb{R}$) by

$$f(x;\alpha,\beta,\mu,\sigma) \quad = \quad \frac{\alpha\,\phi(\frac{x-\mu}{\sigma})\,\Phi(\frac{x-\mu}{\sigma})^{\alpha-1}}{\left[1-\Phi(\frac{x-\mu}{\sigma})\right]^{\alpha+1}} \left\{1+\beta\left[\frac{\Phi(\frac{x-\mu}{\sigma})}{1-\Phi(\frac{x-\mu}{\sigma})}\right]\right\}^{\frac{-1}{\beta}-1}$$

where $\Phi(.), \phi(.)$ denote the cdf and pdf of standard N random variable, respectively. Plots of the pdf of ExOW-N distribution for selected parameter values are shown in Figure 1. Figure 1 shows that the pdf of the ExOW-N is symmetric, bimodal and unimodal. Other shapes can also be obtained using another distribution. These shape properties show that the ExOW-G family can be very useful to fit different data sets with various shapes.

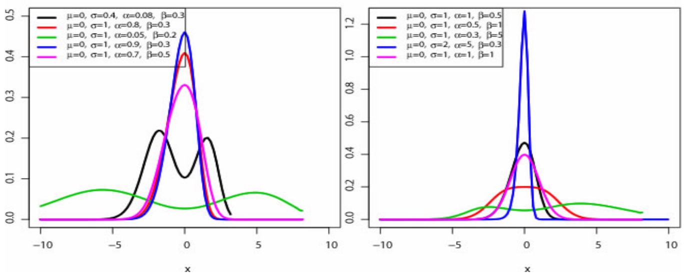

Figure 1. Plots of the pdf of ExOW-N distribution for some parameter values.

3.2. **The ExOW-Weibull (ExOW-W) distribution.** By taking $G(x;\boldsymbol{\xi})$ and $g(x;\boldsymbol{\xi})$ in (1.2) to be the cdf $G(x) = 1 - \mathrm{e}^{-\left(\frac{x}{\theta}\right)^{\lambda}}$ and the pdf $g\left(x\right) = \frac{\lambda}{\theta}\left(\frac{x}{\theta}\right)^{\lambda-1}\exp\left(-(x/\theta)^{\lambda}\right)$ of the Weibull distribution, the pdf of the ExOW-W distribution is given (for $x > 0$) by

$$f(x;\alpha,\beta,\lambda,\theta) = \frac{\alpha\lambda\, x^{\lambda-1}}{\theta^{\lambda}}\,\mathrm{e}^{\alpha\left(\frac{x}{\theta}\right)^{\lambda}}\left[1-\mathrm{e}^{-\left(\frac{x}{\theta}\right)^{\lambda}}\right]^{\alpha-1}\left\{1+\beta\left[\mathrm{e}^{\left(\frac{x}{\theta}\right)^{\lambda}}-1\right]\right\}^{\frac{-1}{\beta}-1}$$

Plots of the pdf and hrf the ExOW-W distribution for selected parameter values are shown in Figure 2. As seen in Figure 2, the distribution has increasing, decreasing and reversed-J shaped hrfs for different values of parameters. This fact implies that the ExOW-W distribution can be very useful for fitting data sets with various shapes.

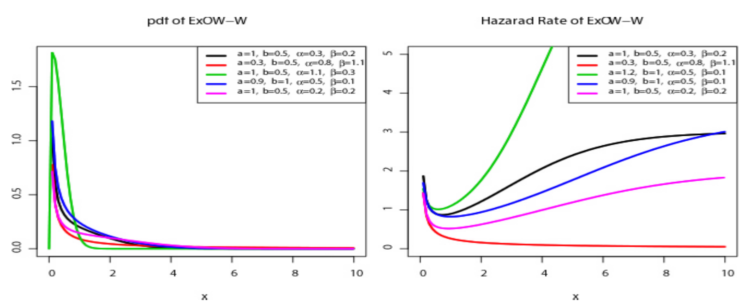

Figure 2. Plots of the pdf and hrf of the ExOW-W distribution for some parameter values.

4. The ExOW-G properties

In this section, we obtain some general properties of the ExOW-G family including asymptotics, ordinary and incomplete moments, quantile and quantile power series, generating function, entropies, order statistics and PWMs.

4.1. **Asymptotics.** Let $d = \inf\{x|G(x) > 0\}$, then, the asymptotics of equations (1.1), (1.2) and (1.3) as $x \to d$ are given by

$$\begin{aligned} F(x) &\sim G(x)^{\alpha} && \text{as} \quad x \to d, \\ f(x) &\sim \alpha g(x)G(x)^{\alpha-1} && \text{as} \quad x \to d, \\ h(x) &\sim \alpha g(x)G(x)^{\alpha-1} && \text{as} \quad x \to d. \end{aligned}$$

The asymptotics of equations (1.1), (1.2) and (1.3) as $x \to \infty$ are given by

$$\begin{aligned} 1 - F(x) &\sim \beta^{\frac{-1}{\beta}} \bar{G}(x)^{\frac{\alpha}{\beta}} && \text{as} \quad x \to \infty, \\ f(x) &\sim \alpha\beta^{\frac{-1}{\beta}-1} g(x)\bar{G}(x)^{\frac{\alpha}{\beta}-1} && \text{as} \quad x \to \infty, \\ h(x) &\sim \frac{\alpha g(x)}{\beta\bar{G}(x)} && \text{as} \quad x \to \infty. \end{aligned}$$

4.2. **Ordinary and incomplete moments.** Let $T_{\alpha k+j}$ denotes the Exp-G distribution with power parameter $\alpha k + j$. The rth moment of X, say μ'_r, follows from (2.5) as

$$\mu'_r = E(X^r) = \sum_{k,j=0}^{\infty} a_{k,j}\ E\left(T^r_{\alpha k+j}\right). \tag{4.1}$$

The nth central moment of X is given by

$$\mu_n = \sum_{r=0}^{n} \binom{n}{r} (-\mu'_1)^{n-r}\ E(X^r) = \sum_{r=0}^{n} \sum_{k,j=0}^{\infty} a_{k,j} \binom{n}{r} (-\mu'_1)^{n-r} E\left(T^r_{\alpha k+j}\right).$$

The cumulants (κ_n) of X follow recursively from

$$\kappa_n = \mu'_n - \sum_{r=0}^{n-1} \binom{n-1}{r-1} \kappa_r\, \mu'_{n-r},$$

where $\kappa_1 = \mu'_1$, $\kappa_2 = \mu'_2 - \mu'^2_1$, $\kappa_3 = \mu'_3 - 3\mu'_2\mu'_1 + \mu'^3_1$, etc. The measures of skewness and kurtosis can be calculated from the ordinary moments using well-known relationships.

The sth incomplete moment of X can be expressed from (2.5) as

$$\varphi_s(t) = \int_{-\infty}^{t} x^s f(x)\, dx = \sum_{k,j=0}^{\infty} a_{k,j} \int_{-\infty}^{t} x^s\, h_{\alpha k+j}(x)\, dx. \tag{4.2}$$

The first incomplete moment, say $\varphi_1(t)$, given by (4.2) with $s = 1$. $\varphi_1(t)$ can be applied to construct Bonferroni and Lorenz curves defined for a given probability π by $B(\pi) = \varphi_1(q)/(\pi\mu'_1)$ and $L(\pi) = \varphi_1(q)/\mu'_1$, respectively, where μ'_1 given by

(4.1) with $r = 1$ and $q = Q(\pi)$ is the qf of X at π. These curves are very useful in economics, reliability, demography, insurance and medicine.

Now, we provide two ways to determine $\varphi_1(t)$. First, a general equation for $\varphi_1(t)$ can be derived from (4.2) as

$$\varphi_1(t) = \sum_{k,j=0}^{\infty} a_{k,j}\; J_{\alpha k+j}(t),$$

where $J_{\alpha k+j}(t) = \int_{-\infty}^{t} x\, h_{\alpha k+j}(x)\, dx$ is the first incomplete moment of the Exp-G family.

A second general formula for $\varphi_1(t)$ is given by

$$\varphi_1(t) = \sum_{k,j=0}^{\infty} a_{k,j}\, v_{\alpha k+j}(t),$$

where $v_{\alpha k+j}(t) = (\alpha k + j)\int_0^{G(t)} Q_G(u)\; u^{\alpha k+j-1} du$ can be computed numerically and $Q_G(u)$ is the quantile function corresponding to $G(x;\boldsymbol{\xi})$, i.e., $Q_G(u) = G^{-1}(u;\boldsymbol{\xi})$.

4.3. **Quantile and generating functions.** The quantile function (qf) of the ExOW-G family follows, by inverting (1.1), as

$$Q(u) = F^{-1}(u) = Q_G\left\{\frac{\left[-1+(1-u)^{-\beta}\right]^{1/\alpha}}{\beta^{1/\alpha}+\left[-1+(1-u)^{-\beta}\right]^{1/\alpha}}\right\}, \tag{4.3}$$

where $Q_G(u) = G^{-1}(u)$ is the qf of the baseline G distribution and $u \in (0, 1)$.

The effects of the shape parameters on the skewness and kurtosis can be based on quantile measures. We obtain skewness and kurtosis measures using the qf. The Bowley's skewness measure is given by

$$Skewness = \frac{Q(1/4) + Q(3/4) - 2Q(1/2)}{Q(3/4) - Q(1/4)}$$

and the Moors's kurtosis measure is

$$Kurtosis = \frac{Q(7/8) - Q(5/8) + Q(3/8) - Q(1/8)}{Q(6/8) - Q(2/8)}.$$

These measures enjoy the advantage of having less sensitivity to outliers. Moreover, they do exist for distribution without moments. Both measures equal zero for the normal distribution. Plots of skewness and kurtosis of the ExOW-N and ExOW-N distributions are presented in Figure 3. These plots indicate that both measures depend very much on the shape parameters and the member of the this family can model various data types in terms of skewness and kurtosis.

Now, we provide two formulae for the moment generating function (mgf) $M_X(t) = E\left(\mathrm{e}^{t\,X}\right)$ of X which can be derived from (2.5).

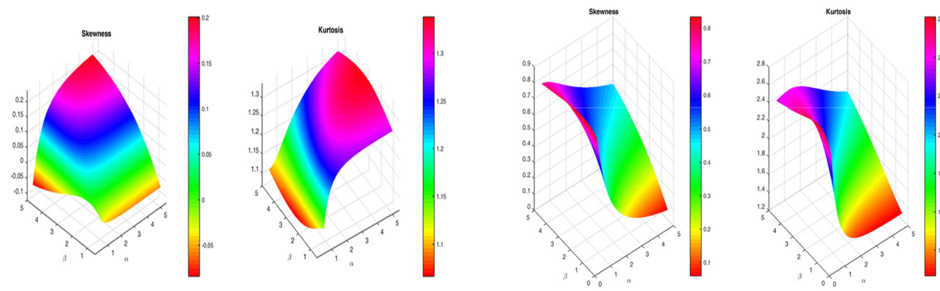

FIGURE 3. Plots of skewness and kurtosis of ExOW-N (left panel) and ExOW-W (right panel) distribution for several values of parameters.

The first one is given by

$$M_X\left(t\right)=\sum_{k,j=0}^{\infty}a_{k,j}\,M_{\alpha k+j}\left(t\right),$$

where $M_{\alpha k+j}\left(t\right)$ is the mgf of $T_{\alpha k+j}$. Hence, $M_X\left(t\right)$ can be determined from the generating function of Exp-G family.

The second formula for $M_X\left(t\right)$ can be expressed as

$$M_X\left(t\right)=\sum_{k,j=0}^{\infty}a_{k,j}\,\tau\left(t,k\right),$$

where $\tau\left(t,k\right)=\int_0^1\exp\left[t\,Q_G\left(u\right)\right]\,u^{\alpha k+j-1}du$.

4.4. **Quantile power series.** In this section, we derive a power series for the qf $x=Q(u)=F^{-1}(u)$ of X by expanding (4.3). If $Q_G(u)$ does not have a closed-form expression, it can be expressed as a power series

$$Q_G(u)=\sum_{i=0}^{\infty}a_i\,u^i, \tag{4.4}$$

where the coefficients $a_i's$ are suitably chosen real numbers. They depend on the parameters of the G distribution. For several important distributions, such as the normal, Student t, gamma and beta distributions, $Q_G(u)$ does not have explicit expressions but it can be expanded as in (4.4). As a simple example, for the N distribution, $a_i=0$ for $i=0,2,4,\ldots$ and $a_1=1$, $a_3=1/6$, $a_5=7/120$ and $a_7=127/7560,\ldots$

We use throughout the paper a result of Gradshteyn and Ryzhik (2000) for a power series raised to a positive integer n (for $n \geq 1$)

$$Q_G(u)^n = \left(\sum_{i=0}^{\infty} a_i\, u^i\right)^n = \sum_{i=0}^{\infty} c_{n,i}\, u^i, \tag{4.5}$$

where $c_{n,0} = a_0^n$ and the coefficients $c_{n,i}$ (for $i = 1, 2, \ldots$) are determined from the recurrence equation

$$c_{n,i} = (i\, a_0)^{-1} \sum_{m=1}^{i} [m(n+1) - i]\, a_m\, c_{n,i-m}. \tag{4.6}$$

Next, we derive an expansion for the argument of $Q_G(\cdot)$ in (4.3), namely

$$A = \frac{\left[-1 + (1-u)^{-\beta}\right]^{\frac{1}{\alpha}}}{\beta^{\frac{1}{\alpha}} + \left[-1 + (1-u)^{-\beta}\right]^{\frac{1}{\alpha}}}.$$

Using the generalized binomial expansion, we have

$$\begin{aligned} \left[-1 + (1-u)^{-\beta}\right]^{\frac{1}{\alpha}} &= (-1)^{\frac{1}{\alpha}} \sum_{i=0}^{\infty} (-1)^i \binom{\frac{1}{\alpha}}{i} (1-u)^{-\beta\, i} \\ &= (-1)^{\frac{1}{\alpha}} \sum_{i,k=0}^{\infty} (-1)^{i+k} \binom{\frac{1}{\alpha}}{i} \binom{-\beta\, i}{k} u^k = \sum_{k=0}^{\infty} \alpha_k^*\, u^k, \end{aligned} \tag{4.7}$$

where

$$\alpha_k^* = \sum_{i,k=0}^{\infty} (-1)^{i+k+\frac{1}{\alpha}} \binom{\frac{1}{\alpha}}{i} \binom{-\beta\, i}{k}$$

and

$$\beta^{\frac{1}{\alpha}} + \left[-1 + (1-u)^{-\beta}\right]^{\frac{1}{\alpha}} = \sum_{k=0}^{\infty} \beta_k^*\, u^k, \tag{4.8}$$

where $\beta_k^* = \alpha_k^* + \beta^{\frac{1}{\alpha}}$ and for $k \geq 1$, $\beta_k^* = \alpha_k^*$. Then, using the ratio of two power series we can write

$$A = \frac{\sum_{k=0}^{\infty} \alpha_k^*\, u^k}{\sum_{k=0}^{\infty} \beta_k^*\, u^k} = \sum_{k=0}^{\infty} \delta_k\, u^k, \tag{4.9}$$

where $\delta_0 = \frac{\alpha_0^*}{\beta_0^*}$ and for $k \geq 1$, we have

$$\delta_k = \frac{1}{\beta_0^*}\left[\alpha_k^* - \frac{1}{\beta_0^*} \sum_{r=1}^{k} \beta_r^* \delta_{k-r}\right]. \tag{4.10}$$

Then, the qf of X can be expressed using (4.3) as

$$Q(u) = Q_G\left(\sum_{k=0}^{\infty} \delta_k\, u^k\right). \tag{4.11}$$

For any baseline G distribution, we combine (4.4) and (4.11) to obtain

$$Q(u)=Q_G\left(\sum_{m=0}^{\infty}\delta_m\,u^m\right)=\sum_{i=0}^{\infty}a_i\,\left(\sum_{m=0}^{\infty}\delta_m\,u^m\right)^i.$$

Then using (4.5) and (4.6), we have

$$Q(u)=\sum_{m=0}^{\infty}e_m\,u^m, \tag{4.12}$$

where $e_m=\sum_{i=0}^{\infty}a_i\,d_{i,m}$, and, for $i=0,1,\ldots$, $d_{i,0}=\delta_0^i$ and (for $m>1$)

$$d_{i,m}=(m\,\delta_0)^{-1}\sum_{n=1}^{m}[n(i+1)-m]\,\delta_n\,d_{i,m-n}.$$

Equation (4.12) reveals that the qf of the ExOW-G family can be expressed as a power series. Then, several mathematical quantities of X can be reduced to integrals over $(0,1)$ based on this power series.

Let $W(\cdot)$ be any integrable function in the real line. We can write

$$\int_{-\infty}^{\infty}W(x)\,f(x)dx=\int_0^1 W\left(\sum_{m=0}^{\infty}e_m\,u^m\right)du. \tag{4.13}$$

Equations (4.12) and (4.13) are the main results of this section since we can obtain various mathematical properties of the ExOW-G family based on them. In fact, they can follow by using the integral on the right-hand side for special $W(\cdot)$ functions, which are usually simple than if they were based on the left-hand integral. For the great majority of these quantities, we can adopt twenty terms in this power series. The formulae derived throughout the paper can be easily handled in most symbolic computation platforms such as Maple, Mathematica and Matlab.

4.5. **Entropies.** The Rényi entropy of a random variable X represents a measure of variation of the uncertainty. The Rényi entropy is given by

$$I_\theta\left(X\right)=\left(1-\theta\right)^{-1}\log\left(\int_{-\infty}^{\infty}f\left(x\right)^{\theta}dx\right),\ \theta>0\text{ and }\theta\neq 1.$$

Using the pdf in (1.2), we can write

$$f\left(x\right)^{\theta}=\frac{\alpha^{\theta}g(x)^{\theta}G(x)^{\theta(\alpha-1)}}{\bar{G}(x)^{\theta(\alpha+1)}}\,\left\{1+\beta\,\left[\frac{G(x)}{\bar{G}(x)}\right]^{\alpha}\right\}^{\frac{-\theta}{\beta}-\theta}.$$

Applying the series expansion (2.1) to the last term, we obtain

$$f\left(x\right)^{\theta}=\alpha^{\theta}g(x)^{\theta}\sum_{k=0}^{\infty}\frac{(\frac{-\theta}{\beta}-\theta)_k}{k!\beta^{-k}}G(x)^{\alpha(k+\theta)-\theta}\left[1-G(x)\right]^{\alpha(\theta-k)+\theta}.$$

Applying the binomial series to the last term, the last equation reduces to

$$f(x)^{\theta} = \alpha^{\theta} g(x)^{\theta} \sum_{k,j=0}^{\infty} (-1)^{j} \frac{(\frac{-\theta}{\beta} - \theta)_{k}}{k!\beta^{-k}} \binom{\alpha(\theta-k)+\theta}{j} G(x)^{\alpha(k+\theta)+j-\theta}.$$

Then, the Rényi entropy of the ExOW-G family is given by

$$I_{\theta}(X) = (1-\theta)^{-1} \log\left[\sum_{k,j=0}^{\infty} \upsilon_{k,j} \int_{-\infty}^{\infty} g(x)^{\theta} G(x)^{\alpha(k+\theta)+j-\theta} dx\right],$$

where

$$\upsilon_{k,j} = \sum_{k,j=0}^{\infty} (-1)^{j} \frac{(\frac{-\theta}{\beta} - \theta)_{k}}{k!\alpha^{-\theta}\beta^{-k}} \binom{\alpha(\theta-k)+\theta}{j}.$$

The θ-entropy can be obtained as

$$H_{\theta}(X) = (1-\theta)^{-1} \log\left[1 - \sum_{k,j=0}^{\infty} \upsilon_{k,j} \int_{-\infty}^{\infty} g(x)^{\theta} G(x)^{\alpha(k+\theta)+j-\theta} dx\right].$$

The Shannon entropy of a random variable X is a special case of the Rényi entropy when $\theta \uparrow 1$. The Shannon entropy, say SI, is defined by $SI = E\{-[\log f(X)]\}$, which follows by taking the limit of $I_{\theta}(X)$ as θ tends to 1.

4.6. **Order statistics.** Order statistics make their appearance in many areas of statistical theory and practice. Let $X_1, \ldots, X_n$ be a random sample from the ExOW-G family. The pdf of $X_{i:n}$ can be written as

$$f_{i:n}(x) = \frac{f(x)}{\mathrm{B}(i, n-i+1)} \sum_{j=0}^{n-i} (-1)^{j} \binom{n-i}{j} F(x)^{j+i-1}, \tag{4.14}$$

where $B(\cdot,\cdot)$ is the beta function. After applying the generalized binomial series to (1.1), we have

$$F(x)^{j+i-1} = \sum_{l=0}^{\infty} (-1)^{l} \binom{j+i-1}{l} \left\{1 + \beta\left[\frac{G(x)}{\bar{G}(x)}\right]^{\alpha}\right\}^{\frac{-l}{\beta}}. \tag{4.15}$$

Using (1.2) and (4.15), we can write

$$f(x) F(x)^{j+i-1} = \frac{\alpha g(x) G(x)^{\alpha-1}}{\bar{G}(x)^{\alpha+1}} \sum_{l=0}^{\infty} (-1)^{l} \binom{j+i-1}{l} \left\{1 + \beta\left[\frac{G(x)}{\bar{G}(x)}\right]^{\alpha}\right\}^{\frac{-(l+1)}{\beta}-1}.$$

From the series expansion (2.1), we get

$$f(x) F(x)^{j+i-1} = \alpha g(x) \sum_{l,k=0}^{\infty} (-1)^{l} \frac{(\frac{-l-1}{\beta} - 1)_{k}}{k!\beta^{-k}} \binom{j+i-1}{l} \frac{G(x)^{\alpha(k+1)-1}}{\bar{G}(x)^{\alpha(k+1)+1}}.$$

Applying the power series (2.3), and after some algebra, we obtain

$$f(x)\,F(x)^{j+i-1} = \alpha \sum_{l,k,s=0}^{\infty} \frac{(-1)^{l}\,(\frac{-l-1}{\beta}-1)_{k}\Gamma(\alpha[k+1]+s+1)}{k!s!\beta^{-k}\Gamma(\alpha[k+1]+1)} \times \binom{j+i-1}{l} g(x)G(x)^{\alpha(k+1)+s-1}. \quad (4.16)$$

Substituting (4.16) in (4.14), the pdf of $X_{i:n}$ can be expressed as

$$f_{i:n}(x) = \sum_{k,s=0}^{\infty} d_{k,s}\; h_{\alpha(k+1)+s}(x),$$

where $h_{\alpha(k+1)+s}(x)$ is the Exp-G density with power parameter $\alpha(k+1)+s$ and

$$d_{k,s} = \sum_{j=0}^{n-i}\sum_{l=0}^{\infty} \frac{\alpha\,(-1)^{l+j}\,(\frac{-l-1}{\beta}-1)_{k}\Gamma(\alpha[k+1]+s)}{k!s!\beta^{-k}\mathrm{B}(i,n-i+1)\,\Gamma(\alpha[k+1]+1)} \binom{n-i}{j}\binom{j+i-1}{l}.$$

Then, the density function of the ExOW-G order statistics is a mixture of Exp-G densities. Based on the last equation, we note that the properties of $X_{i:n}$ follow from those properties of $T_{\alpha(k+1)+s}$.

The qth moments of $X_{i:n}$ can be expressed as

$$E\left(X_{i:n}^{q}\right) = \sum_{k,s=0}^{\infty} d_{k,s}\, E\left(T_{\alpha(k+1)+s}\right). \quad (4.17)$$

Based upon the moments in (4.17), we can derive explicit expressions for the L-moments of X as infinite weighted linear combinations of the means of suitable ExOW-G order statistics. The rth L-moments is given by

$$\lambda_r = \frac{1}{r}\sum_{d=0}^{r-1}(-1)^{d}\binom{r-1}{d} E\left(X_{r-d:r}\right),\; r \geq 1.$$

4.7. **PWMs.** The PWM is the expectation of certain function of a random variable whose mean exists. A general theory for the PWMs covers the summarization and description of theoretical probability distributions and observed data samples, nonparametric estimation of the underlying distribution of an observed sample, estimation of parameters, quantiles of probability distributions and hypothesis tests. The PWM method can generally be used for estimating parameters of a distribution whose inverse form cannot be expressed explicitly.

The (j,i)th PWM of X following the ExOW-G family, say $\rho_{j,i}$, is formally defined by

$$\rho_{j,i} = E\left\{X^{j}F(X)^{i}\right\} = \int_{-\infty}^{\infty} x^{j} f(x)\,F(X)^{i}\,dx.$$

From (4.16), we can write

$$\begin{aligned} f(x)\,F(x)^{i} &= \alpha \sum_{l,k,s=0}^{\infty} \frac{(-1)^{l}\,(\frac{-l-1}{\beta}-1)_{k}\Gamma\left(\alpha\left[k+1\right]+s+1\right)}{k!s!\beta^{-k}\Gamma\left(\alpha\left[k+1\right]+1\right)} \\ &\quad \times \binom{i}{l} g(x)G(x)^{\alpha(k+1)+s-1}. \end{aligned}$$

The last equation can be expressed as

$$f(x)\,F(X)^{i} = \sum_{k,s=0}^{\infty} m_{k,s}\; h_{\alpha(k+1)+s}(x)\,,$$

where

$$m_{k,s} = \sum_{l=0}^{\infty} \frac{\alpha\,(-1)^{l}\,(\frac{-l-1}{\beta}-1)_{k}\Gamma\left(\alpha\left[k+1\right]+s\right)}{k!s!\beta^{-k}\Gamma\left(\alpha\left[k+1\right]+1\right)}\binom{i}{l}.$$

Then, the PWM of X is given by

$$\rho_{j,i} = \sum_{k,s=0}^{\infty} m_{k,s} \int_{-\infty}^{\infty} x^{j}\; h_{\alpha(k+1)+s}(x)\,dx = \sum_{k,s=0}^{\infty} m_{k,s}\; E\left(T^{j}_{\alpha(k+1)+s}\right).$$

5. Maximum likelihood estimation

Several approaches for parameter estimation were proposed in the literature but the maximum likelihood method is the most commonly employed. The MLEs enjoy desirable properties and can be used to obtain confidence intervals for the model parameters. The normal approximation for these estimators in large samples can be easily handled either analytically or numerically. Here, we consider the estimation of the unknown parameters of the new family from complete samples only by maximum likelihood.

Let $X_1, ..., X_n$ be a random sample from the ExOW-G family with parameters α, β and $\boldsymbol{\xi}$. Let $\boldsymbol{\theta} = (\alpha, \beta, \boldsymbol{\xi}^{\intercal})^{\intercal}$ be the $p \times 1$ parameter vector. To obtain the MLE of $\boldsymbol{\theta}$, the log-likelihood function is given by

$$\begin{aligned} \ell(\boldsymbol{\theta}) &= n\log(\alpha) + \sum_{i=1}^{n} \log g\left(x_i; \boldsymbol{\xi}\right) - (\alpha+1)\sum_{i=1}^{n} \log\left[1 - G(x_i;\boldsymbol{\xi})^{\alpha}\right] \\ &\quad + (\alpha-1)\sum_{i=1}^{n} \log G\left(x_i;\boldsymbol{\xi}\right) - \left(\frac{1}{\beta}+1\right)\sum_{i=1}^{n} \log\left[1 + \frac{\beta G(x_i;\boldsymbol{\xi})^{\alpha}}{\bar{G}(x_i;\boldsymbol{\xi})^{\alpha}}\right]. \end{aligned}$$

Then, the score vector components, $\mathbf{U}(\boldsymbol{\theta}) = \frac{\partial \ell}{\partial \boldsymbol{\theta}} = \left(U_{\alpha}, U_{\beta}, U_{\boldsymbol{\xi}_k}\right)^{\intercal}$, are

$$
\begin{aligned}
U_{\alpha} &= \frac{n}{\alpha}-\sum_{i=1}^{n}\log\left[1-G(x_i;\boldsymbol{\xi})^{\alpha}\right]+\alpha\sum_{i=1}^{n}\frac{G(x_i;\boldsymbol{\xi})^{\alpha}\log G\left(x_i;\boldsymbol{\xi}\right)}{1-G(x_i;\boldsymbol{\xi})^{\alpha}} \\
&\quad+\sum_{i=1}^{n}\log G\left(x_i;\boldsymbol{\xi}\right)-(1+\beta)\sum_{i=1}^{n}\frac{G(x_i;\boldsymbol{\xi})^{\alpha}\bar{G}(x_i;\boldsymbol{\xi})^{\alpha}\log\left[\frac{G(x_i;\boldsymbol{\xi})}{\bar{G}(x_i;\boldsymbol{\xi})}\right]}{\bar{G}(x_i;\boldsymbol{\xi})^{2\alpha}+\beta G(x_i;\boldsymbol{\xi})^{\alpha}\bar{G}(x_i;\boldsymbol{\xi})^{\alpha}},
\end{aligned}
$$

$$
U_{\beta}=\frac{1}{\beta^{2}}\sum_{i=1}^{n}\log\left[1+\frac{\beta G(x_i;\boldsymbol{\xi})^{\alpha}}{\bar{G}(x_i;\boldsymbol{\xi})^{\alpha}}\right]-\frac{1}{\beta}\sum_{i=1}^{n}\frac{G(x_i;\boldsymbol{\xi})^{\alpha}}{\beta G(x_i;\boldsymbol{\xi})^{\alpha}+\bar{G}(x_i;\boldsymbol{\xi})^{\alpha}}
$$

and

$$
\begin{aligned}
U_{\boldsymbol{\xi}_k} &= \sum_{i=1}^{n}\frac{g'\left(x_i;\boldsymbol{\xi}\right)}{g\left(x_i;\boldsymbol{\xi}\right)}+(\alpha+1)\sum_{i=1}^{n}\frac{\alpha G'(x_i;\boldsymbol{\xi})^{\alpha-1}}{1-G(x_i;\boldsymbol{\xi})^{\alpha}}+(\alpha-1)\sum_{i=1}^{n}\frac{G'\left(x_i;\boldsymbol{\xi}\right)}{G\left(x_i;\boldsymbol{\xi}\right)} \\
&\quad-\alpha\left(\beta+1\right)\sum_{i=1}^{n}\frac{\bar{G}(x_i;\boldsymbol{\xi})^{\alpha}G'(x_i;\boldsymbol{\xi})^{\alpha-1}-G(x_i;\boldsymbol{\xi})^{\alpha}\bar{G}'(x_i;\boldsymbol{\xi})^{\alpha-1}}{\bar{G}(x_i;\boldsymbol{\xi})^{\alpha}\left[\beta G(x_i;\boldsymbol{\xi})^{\alpha}+\bar{G}(x_i;\boldsymbol{\xi})^{\alpha}\right]},
\end{aligned}
$$

where $g'\left(x_i;\boldsymbol{\xi}\right)=\partial g\left(x_i;\boldsymbol{\xi}\right)/\partial\boldsymbol{\xi}_k$ and $G'\left(x_i;\boldsymbol{\xi}\right)=\partial G\left(x_i;\boldsymbol{\xi}\right)/\partial\boldsymbol{\xi}_k$.

Setting the nonlinear system of equations $U_{\alpha}=U_{\beta}=0$ and $U_{\boldsymbol{\xi}_{\mathbf{k}}}=\mathbf{0}$ and solving them simultaneously yields the MLE $\widehat{\boldsymbol{\theta}}=(\widehat{\alpha},\widehat{\beta},\widehat{\boldsymbol{\xi}}^{\intercal})^{\intercal}$. For doing this, it is usually more convenient to adopt nonlinear optimization methods such as the quasi-Newton algorithm to maximize ℓ numerically. For interval estimation of the parameters, we obtain the $p\times p$ observed information matrix $J(\boldsymbol{\theta})=\{\frac{\partial^2\ell}{\partial r\,\partial s}\}$ (for $r,s=\alpha,\beta,\boldsymbol{\xi}$), whose elements can be computed numerically.

Under standard regularity conditions when $n\to\infty$, the distribution of $\widehat{\boldsymbol{\theta}}$ can be approximated by a multivariate normal $N_p(0,J(\widehat{\boldsymbol{\theta}})^{-1})$ distribution to obtain confidence intervals for the parameters. Here, $J(\widehat{\boldsymbol{\theta}})$ is the total observed information matrix evaluated at $\widehat{\boldsymbol{\theta}}$. The method of the re-sampling bootstrap can be used for correcting the biases of the MLEs of the model parameters. Good interval estimates may also be obtained using the bootstrap percentile method. Improved MLEs can be obtained for the new family using second-order bias corrections. However, these corrected estimates depend on cumulants of log-likelihood derivatives and will be addressed in future research.

6. Simulation

6.1. **Simulation for the ExOW-N distribution.** In this subsection, a simulation study is conducted to examine the performance of the MLEs of the ExOW-N parameters. We generate 1000 samples of size, n =50, 100, 250, 500 and 1000 of the ExOW-N model. The precision of the MLEs is discussed by means of the following measures: mean, mean square error (MSE), estimated average length (AL) and coverage probability (CP). The empirical study was conducted with software R. The empirical results are given in Table 1. The values in Table 1 indicate that the estimates are quite stable and, more importantly, are close to the true values for the these sample sizes. The simulation study shows that the maximum likelihood method is appropriate for estimating the ExOW-N parameters. In fact, the means of the parameters tend to be closer to the true parameter values when n increases. This fact supports that the asymptotic normal distribution provides an adequate approximation to the finite sample distribution of the MLEs. The normal approximation can be improved by using bias adjustments to these estimators. Approximations to the their biases in simple models may be obtained analytically.

TABLE 1. Simulation results of the ExOW-N distribution for several values of parameters.

$\mu=0.5,\sigma=2$			Mean				MSE				AL				CP			
α	β	n	α	β	μ	σ	α	β	μ	σ	α	β	μ	σ	α	β	μ	σ
0.5	0.5	50	0.531	0.482	0.646	2.018	0.163	0.429	0.760	1.049	1.906	3.209	4.884	5.491	0.843	0.913	0.933	0.859
		100	0.490	0.491	0.563	1.978	0.058	0.193	0.384	0.452	1.102	2.333	3.577	3.343	0.886	0.942	0.951	0.921
		250	0.491	0.486	0.553	1.974	0.021	0.062	0.152	0.182	0.598	1.204	1.888	1.803	0.913	0.941	0.955	0.922
		500	0.497	0.492	0.519	1.997	0.012	0.036	0.091	0.106	0.410	0.790	1.252	1.255	0.924	0.945	0.954	0.937
		1000	0.501	0.499	0.511	2.008	0.005	0.018	0.047	0.046	0.291	0.545	0.874	0.885	0.946	0.948	0.952	0.954
2	1.5	50	2.041	1.901	0.473	1.871	1.991	1.620	0.114	1.536	9.038	4.900	1.347	8.110	0.813	0.926	0.918	0.791
		100	1.997	1.667	0.492	1.933	1.144	0.668	0.054	0.946	6.503	2.883	0.909	6,048	0.845	0.933	0.945	0.847
		250	1.996	1.578	0.491	1.969	0.729	0.179	0.019	0.646	4.199	1.598	0.543	3.993	0.896	0.952	0.952	0.888
		500	2.072	1.532	0.498	2.053	0.423	0.071	0.007	0.377	3.286	1.044	0.368	3.153	0.927	0.948	0.956	0.922
		1000	2.040	1.518	0.497	2.034	0.253	0.035	0.004	0.229	2.226	0.725	0.259	2.144	0.945	0.952	0.954	0.941
1.5	0.8	50	1.390	0.874	0.542	1.799	0.898	0.672	0.142	1.174	6.230	2.980	1.499	7.333	0.830	0.894	0.928	0.811
		100	1.448	0.837	0.528	1.911	0.607	0.234	0.064	0.81	4.515	1.735	0.953	5.420	0.835	0.925	0.947	0.845
		250	1.542	0.824	0.502	2.034	0.389	0.067	0.018	0.545	3.247	0.984	0.559	3.962	0.888	0.944	0.946	0.886
		500	1.497	0.817	0.496	1.989	0.18	0.031	0.009	0.269	2.111	0.666	0.385	2.591	0.924	0.944	0.948	0.926
		1000	1.511	0.807	0.493	2.006	0.116	0.013	0.005	0.173	1.579	0.463	0.269	1.539	0.946	0.95	0.952	0.942
3	2	50	2.636	2.402	0.500	1.725	2.693	2.135	0.052	1.017	5.804	5.849	0.999	4.997	0.863	0.915	0.940	0.809
		100	2.744	2.326	0.474	1.761	1.664	0.959	0.029	0.662	3.584	3.643	0.672	3.703	0.932	0.944	0.946	0.884
		250	2.906	2.111	0.490	1.841	1.005	0.244	0.009	0.426	1.301	1.934	0.398	2.441	0.934	0.952	0.947	0.908
		500	2.951	2.032	0.495	1.929	0.617	0.103	0.004	0.245	0.968	1.310	0.278	1.330	0.946	0.952	0.948	0.944
		1000	2.991	2.016	0.499	1.998	0.346	0.059	0.002	0.155	0.467	0.504	0.193	0.756	0.949	0.950	0.948	0.949

6.2. **Simulation Results for the ExOW-W distribution.** We assess the performance of the MLEs of the ExOW-W parameters by means of two simulation studies. The precision of the MLEs is discussed by means of the following measures: bias, MSE, AL and CP. We generate $N = 1000$ samples of size $n = 50, 55, \ldots, 1000$ from the ExOW-W distribution with $\alpha = 0.5, \beta = 0.5, \lambda = 0.5, \theta = 2$ using the inverse transform method. The MLEs of the parameters are obtained for each generated sample, say $(\hat{\alpha_i}, \hat{\beta_i}, \hat{\lambda_i}, \hat{\theta_i})$, for $i = 1, 2, \cdots, N$. The standard errors of the MLEs are evaluated by inverting the observed information matrix, namely $(s_{\hat{\alpha_i}}, s_{\hat{\beta}_i}, s_{\hat{\lambda_i}}, s_{\hat{\theta_i}})$ for $i = 1, 2, \cdots, N$. The estimated biases and MSEs are given by

$$\widehat{Bias}_{\epsilon}(n) = \frac{1}{N}\sum_{i=1}^{N}(\hat{\epsilon_i} - \epsilon),$$

$$\widehat{MSE}_{\epsilon}(n) = \frac{1}{N}\sum_{i=1}^{N}(\hat{\epsilon_i} - \epsilon)^2,$$

for $\epsilon = \alpha, \beta, \lambda, \theta$. The CPs and ALs are, respectively, given by

$$CP_\epsilon(n) = \frac{1}{N}\sum_{i=1}^{N} I(\hat{\epsilon_i} - 1.95996 s_{\hat{\epsilon_i}}, \hat{\epsilon_i} + 1.95996 s_{\hat{\epsilon_i}}),$$

$$AL_\epsilon(n) = \frac{3.919928}{N}\sum_{i=1}^{N} s_{\hat{\epsilon_i}}.$$

The numerical results for the above measures are displayed in Figures 4 to 7. We note from these plots that the estimated biases decrease as the sample size n increases. Further, the estimated MSEs decay toward zero as n increases. This fact reveals the consistency property of the MLEs. The CP is near to 0.95 and approaches to the nominal value when the sample size increases. Moreover, if the sample size increases, the AL decreases in each case. The reported results are obtained for a selected parameter vector $(\alpha, \beta, \lambda, \theta)$. However, similar results are hold for several parameter values.

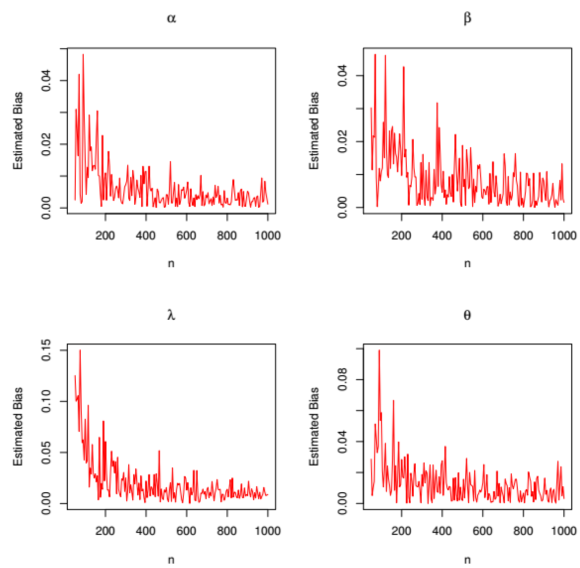

FIGURE 4. Estimated bias of the selected parameter vector.

7. APPLICATION

In this section, we provide two applications to real data sets to illustrate the flexibility of the ExOW-G family. In each case, the parameters are estimated by maximum likelihood method and R statistical software is used for computations.

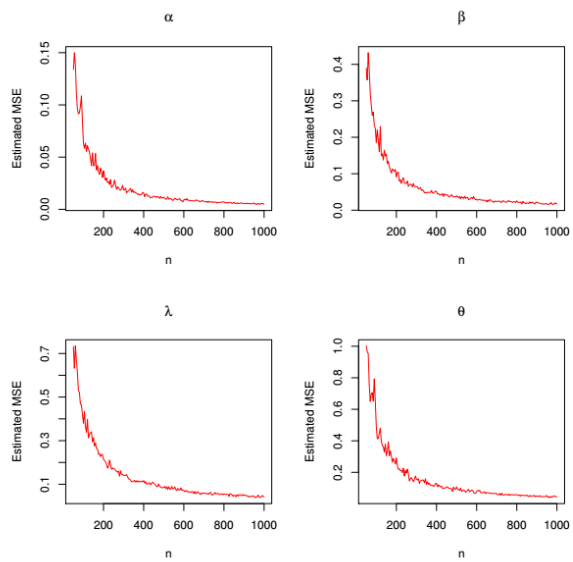

FIGURE 5. Estimated MSEs of the selected parameter vector.

First, we describe the data sets and then determine the MLEs (and the corresponding standard errors) of the parameters. In order to compare the above mentioned models with the proposed family, we apply goodness-of-fit tests to verify which distribution fits better the real data set. The statistics Cramer von Mises (W*) and Anderson Darling (A*) are described in details in Chen and Balakrishnan (1995). The log-likelihood values are also obtained for all models and used to decide best model. In general, the smaller the values of these statistics, the better the fit to the data.

7.1. **First Application with the ExOW-W distribution.** The first real data set was originally reported by Proschan (1963), which consists of 213 observations on the number of successive failures of the air conditioning system of a fleet of 13 Boeing 720 jet airplanes. The data are as follows:

50, 130, 487, 57, 102, 15, 14, 10, 57, 320, 261, 51, 44, 9, 254, 493, 33, 18, 209, 41, 58, 60, 48, 56, 87, 11, 102, 12, 5, 14, 14, 29, 37, 186, 29, 104, 7, 4, 72, 270, 283, 7, 61, 100, 61, 502, 220, 120, 141, 22, 603, 35, 98, 54, 100, 11, 181, 65, 49, 12, 239, 14, 18, 39, 3, 12, 5, 32, 9, 438, 43, 134, 184, 20, 386, 182, 71, 80, 188, 230, 152, 5, 36, 79, 59, 33, 246, 1, 79, 3, 27, 201, 84, 27, 156, 21, 16, 88, 130, 14, 118, 44, 15, 42, 106, 46, 230, 26, 59, 153, 104, 20, 206, 5, 66, 34, 29, 26, 35, 5, 82, 31, 118, 326, 12, 54, 36, 34, 18, 25, 120, 31, 22, 18, 216, 139, 67, 310, 3, 46, 210, 57, 76, 14, 111, 97, 62, 39, 30, 7, 44, 11, 63, 23, 22, 23, 14, 18, 13, 34, 16, 18, 130, 90, 163, 208, 1, 24, 70, 16, 101, 52, 208, 95, 62, 11, 191, 14, 71.

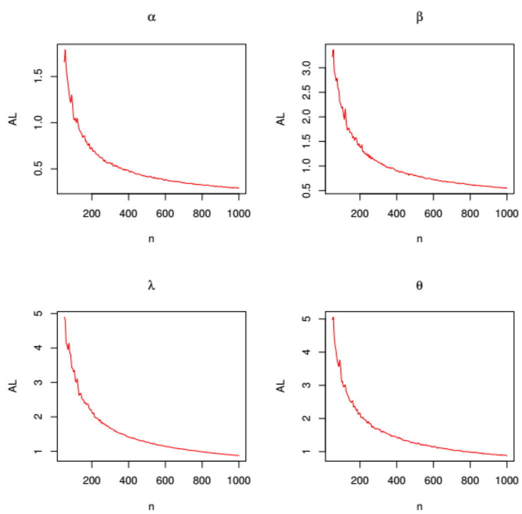

FIGURE 6. Estimated ALs of the selected parameter vector.

FIGURE 6. Estimated ALs of the selected parameter vector.

The descriptive statistics for the first data are: $\bar{x} = 89.134$, $\sigma^2 = 11142.35$, $skewness = 2.233$ and $kurtosis = 8.735$. According to these results, the data is right-skewed and leptokurtic. Now, we compare the fitting performance of ExOW-W with other known models, namely Weibull (W), odd log-logistic-Weibull (OLL-W), generalized odd log-logistic-Weibull (GOLL-W), Kumaraswamy-Weibull (Kum-W), exponentiated generalized-Weibull (EG-W), odd Burr-Weibull (OBu-W), and Weibull-Weibull (W-W). The fitted distributions and their abbreviations are presented in Table 2.

TABLE 2. Fitted distributions and their abbreviations for the first data set.

Distributions	References
Weibull	Weibull (1951)
Odd Log-Logistic-Weibull	Gleaton and Lynch (2006)
Generalized Odd Log-Logistic-Weibull	Cordeiro *et al.* (2016)
Kumaraswamy-Weibull	Cordeiro *et al.* (2010)
Exponentiated Generalized-Weibull	Oguntunde *et al.* (2015)
Odd Burr-Weibull	Alizadeh *et al.* (2017)
Weibull-Weibull	Bourguignon *et al.* (2014)
ExOW-W	Proposed

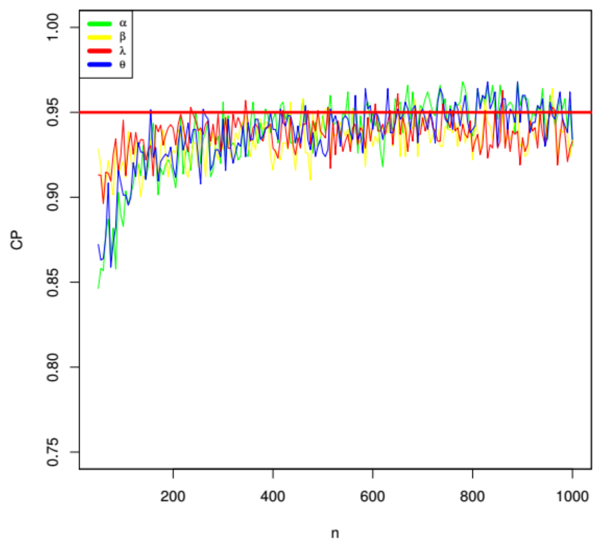

FIGURE 7. Estimated CPs of the selected parameter vector.

Table 3 gives W* and A* statistics and log-likelihood values. Based on Table 3, it is clear that the ExOW-W distribution provides the overall best fit and therefore could be chosen as the more adequate model from other models for explaining the first data set.

More information can be provided in Figure 8 by a histogram of the data with fitted lines of the pdfs for all distributions. Figure 8 suggests that the ExOW-W fits skewed data very well. Then, we present the plots of the fitted density, cumulative and survival functions with the probability-probability (P-P) plot for the ExOW-W model in Figure 9. They reveal a good adjustment for the data of the estimated density, cumulative and survival functions of the ExOW-W distribution.

7.2. **Second Application with the ExOW-N distribution.** The second data set is related to failure data of Hong and Meeker (2013) in weeks of a product called Product D2 that is used in offices or residences. Gitifar *et al.* (2016) selected one hundred data randomly from a total of 1800 observations to evaluate the fitting performance of their compounding distributions. The selected data are as follows:

Table 3. Fitting summary of distributions for the first data set.

Models	α	β	λ	θ	A^*	W^*	$-\ell$
W			0.906	84.697	0.975	0.157	981.147
			0.051	7.390			
OLL-W	1.633		0.600	96.282	0.572	0.086	979.418
	0.572		0.193	17.272			
GOLL-W	0.573	5.360	0.661	16.560	0.198	0.025	976.603
	0.201	2.442	0.146	7.914			
Kum-W	2.963	0.154	0.732	4.974	0.218	0.028	976.779
	0.188	0.012	0.007	0.009			
EG-W	1.044	3.353	0.509	20.008	0.289	0.039	977.565
	3.807	1.804	0.119	141.242			
OBu-W	1.771	0.154	0.794	14.988	0.166	0.020	976.387
	0.341	0.082	0.072	6.602			
W-W	0.023	5.056	0.107	27.097	0.340	0.218	983.643
	0.029	1.901	0.036	37.114			
ExOW-W	1.618	4.254	0.95	39.557	0.158	0.019	976.178
	0.326	1.889	0.103	11.161			

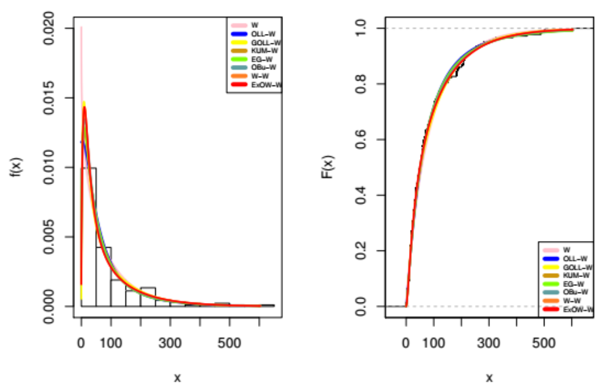

Figure 8. Fitted densities of distributions for the first data set.

0.222673061 0.257639905 0.328155859 0.515672484 0.583401130 0.642256077 0.621521735 0.587506929 0.594755485 0.316753044 0.550884304 0.312962380 0.516646945 0.546445582 0.600493703 0.297813235 0.332441913 0.333245894 0.364800151 0.429097225 0.627439232 0.313363071 0.579554283 0.391397547 0.125167305 0.541816854 0.665764686 0.398880874 0.402492151 0.423982077 0.428143776 0.341767913 0.514537781 0.686683383 0.333088363 0.249962985 0.226748439 0.286643595 0.645490088 0.584664074 0.397377064 0.609634794 0.353187577 0.536304985 0.406031202 0.586163204 0.648786836 0.516497130 0.318475607 0.494774308 0.436782434 0.245923132 0.618409876 0.255245760 0.464312202 0.454133994 0.387982016 0.218311879 0.526363495 0.418258490 0.272839591 0.151997829 0.492728139 0.290973052 0.471553883 0.363069573 0.668371780 0.501805967 0.600306622 0.477109810 0.515188714 0.283784543 0.600625759 0.299420135 0.368553098 0.653382502 0.687845701 0.379423961 0.279504337 0.407995757 0.685695223 0.259685231 0.514854899 0.501119729 0.003522425 0.672089253 0.630145059 0.310811342 0.384073475 0.388312955 0.268080935 0.437408445 0.634243302 0.239656858 0.391844012 0.347107733 0.499160234 0.325770026 0.290634387 0.371908794.

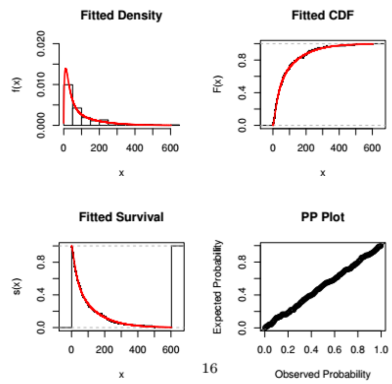

FIGURE 9. Plots for fitted functions of the ExOW-W model for the first data set.

The descriptive statistics for second data are: $\bar{x} = 0.435$, $\sigma^2 = 0.0219$, $skewness = -0.126$ and $kurtosis = 2.386$. These results show that the data is left-skewed and leptokurtic. Since there are not many flexible distributions to model left-skewed data in statistics literature, we evaluate the fitting performance of the ExOW-N distribution in modelling left skewed data. The fitting performance of the ExOW-N distribution is compared with other known models, namely Normal (N), odd log-logistic-normal (OLL-N), generalized odd log-logistic-normal (GOLL-N), Kumaraswamy-normal (Kum-N), exponentiated generalized-normal (EG-N), odd Burr-normal (OBu-N), and Weibull-normal (W-N) distributions given in Table 4.

TABLE 4. Fitted distributions and their abbreviations for the second data set.

Distributions	References
Normal	Gauss (1809)
Odd Log-Logistic-Normal	Braga *et al.* (2016)
Generalized odd log-logistic-Normal	Cordeiro *et al.* (2016)
Kumaraswamy-Normal	Cordeiro and de Castro (2011)
Exponentiated Generalized-Normal	Cordeiro *et al.* (2013)
Weibull-Normal	Bourguignon *et al.* (2014)
Odd Burr-Normal	Alizadeh *et al.* (2017)
ExOW-N	Proposed

Table 5 gives W* and A* statistics and log-likelihood values. Table 5 shows that the ExOW-N distribution provides the overall best fit and therefore could be chosen as the more adequate model from other models for explaining the second data set.

Table 5. Fitting summary of distributions for the second data set

Models	α	β	μ	σ	A^*	W^*	$-\ell$
N			0.435	0.147	0.894	0.135	-49.457
			0.014	0.010			
OLL-N	0.235		0.442	0.051	0.337	0.034	-53.449
	0.112		0.011	0.015			
GOLL-N	0.245	0.410	0.497	0.039	0.43	0.06	-54.272
	0.119	0.311	0.046	0.011			
Kum-N	17.626	12.869	-0.219	0.648	1.063	0.169	-49.353
	44.647	53.755	1.431	0.947			
EG-N	0.625	0.162	0.583	0.072	0.943	0.149	-50.515
	0.005	0.0173	0.004	0.001			
OBu-N	0.201	1.297	0.467	0.048	0.285	0.026	-54.203
	0.096	0.260	0.022	0.014			
W-N	0.989	0.127	0.496	0.045	0.1805	0.019	-57.248
	0.175	0.056	0.016	0.012			
ExOW-N	0.127	0.009	0.498	0.046	0.177	0.018	-57.276
	0.064	0.515	0.041	0.012			

Figure 10 shows a histogram of the data with fitted lines of the pdfs for all distributions. As seen in Figure 10, the ExOW-N distribution fits left-skewed data very well. Then, the plots of the fitted density, cumulative and survival functions with the P-P plot for the ExOW-N model are presented in Figure 11. They reveal a good adjustment for the data of the estimated density, cumulative and survival functions of the ExOW-N distribution.

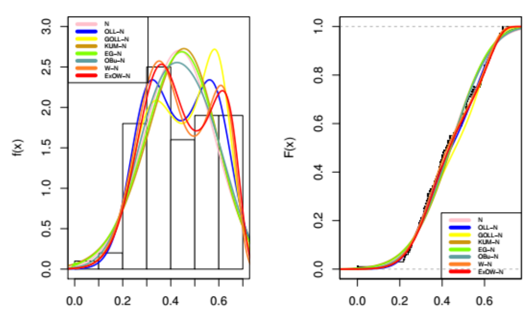

Figure 10. Fitted densities of distributions for the second data set.

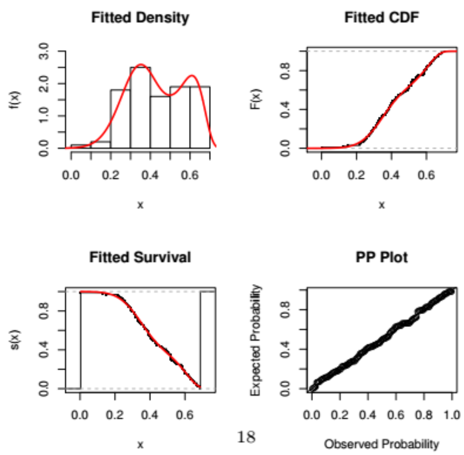

Figure 11. Plots for fitted functions of the ExOW-N model for the second data set.

8. Conclusions

The Weibull distribution is one of the most widely used lifetime distributions in reliability. However, a drawback of this distribution as far as lifetime analysis is concerned is the monotonic behaviour of its hazard rate function (hrf). In real life applications, empirical hazard rate curves often exhibit non-monotonic shapes such as a bathtub, upside-down bathtub (unimodal) and others. Hence, there is a genuine desire to search for some generalizations or modifications of the Weibull distribution that can provide more flexibility in lifetime modeling. There is great interest among statisticians and practitioners in the past decade to generate new and flexible generalized families from classic ones. We have presented a new *extended odd Weibull-G* (ExOW-G) family by adding two extra shape parameters. Many well-known distributions emerge as special cases of the ExOW-G family. The ordinary Weibull-G family is particular case of the proposed distribution family. The mathematical properties of the new family including explicit expansions for the ordinary, probability weighted and incomplete moments, quantile and generating functions, entropies, order statistics have been provided. The model parameters have been estimated by the maximum likelihood estimation method. It is shown, by means of two real data sets, that special cases of the ExOW-G family can provide better fits than other families each having the same number of parameters. Further, we obtain that this family could generate a bimodal shaped distribution. Finally we conclude that adding parameters to any continuous distribution via ExOW-G family construction increases its flexibility. We hope that the ExOW-G family may be extensively used in statistics.

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