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Research Article

A Study on Trans-para-Sasakian Manifolds

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In memory of Professor Simeon Zamkovoy

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Abstract

In the current paper, we make the first contribution to investigate under which conditions three-dimensional trans-para-Sasakian manifold has η-parallel Ricci tensor and cyclic parallel Ricci tensor. Finally, a three-dimensional trans-para-Sasakian manifold example that satisfies our results is constructed.

Keywords: Trans-para-Sasakian manifold, η-parallel Ricci tensor, Cyclic Ricci tensor.

1. Introduction

Two new kinds of almost contact structures are called trans-Sasakian and almost trans-Sasakian structures introduced by Oubina [1]. Then, Blair and Oubina [2] gave a condition for a structure (ϕ,ζ,η,g) to being trans- Sasakian structure as follows.

$$(\nabla_{\vartheta_1}\phi)\vartheta_2=\alpha[g(\vartheta_1,\vartheta_2)\zeta-\eta(\vartheta_2)\vartheta_1]+\beta[g(\phi\vartheta_1,\vartheta_2)\zeta-\eta(\vartheta_2)\phi\vartheta_1]. \quad (1)$$

Trans-Sasakian manifolds have arisen naturally out of the classification of almost contact metric structures by Chinea and Gonzales [3]. Marrero completely characterized trans-Sasakian manifolds of dimension $n \geq 5$ [4].

In [5], Zamkovoy introduced the trans-para-Sasakian manifolds (shortly tpS) and studied some curvature properties. A tpS manifold has a tpS structure of type (α,β), where α and β are smooth functions. Özkan et al. [6] studied the geometry of tpS manifolds. This study is prepared as follows. In Section 2, we present some properties of $(2n + 1)$-dimensional tpS manifolds. In Section 3, we prove that a 3-dimensional tpS manifold for α and β are constants has η- parallel Ricci tensor iff the manifold has constant scalar curvature. In the last section, we show that a 3-dimensional tpS manifold for α and β are constants which is not α-para-Sasakian manifold satisfies cyclic parallel Ricci tensor iff the scalar curvature $r=-6(\alpha^2+\beta^2-\zeta(\beta))$. Finally, a three-dimensional tpS manifold example that satisfies our results is constructed.

2. Material and Methods

M^{2n+1} is called an almost paracontact manifold if it has (ϕ,ζ,η) such that the followings hold

$$\eta(\zeta)=1,\ \phi^2=I-\eta\otimes\zeta \quad (2)$$

and $\mathcal{D}=\ker(\eta)$, where ϕ,ξ and η are (1,1)-tensor field, vector field and 1-form, resp. As a natural consequence, the tensor field ϕ has rank $2n, \phi\zeta=0$ and $\eta\circ\phi=0$. Here, ζ denotes a certain vector field which is dual to η and satisfying $d\eta(\zeta,\vartheta_1)=0$ for all $\vartheta_1\in\chi(M)$. Within the framework of almost paracontact manifolds, if the tensor field $N_\phi:=[\phi,\phi]-2d\eta\otimes\zeta=0$, then the almost paracontact manifold is called normal [7]. If (M,ϕ,ζ,η) has a pseudo-Riemannian metric such that

$$g(\phi\vartheta_1,\phi\vartheta_2)=-g(\vartheta_1,\vartheta_2)+\eta(\vartheta_1)\eta(\vartheta_2), \quad (3)$$

then we say that (M,ϕ,ζ,η,g) is an almost paracontact metric manifold. The signature of the pseudo-Riemannian metric is $(n+1,n)$. An orthogonal basis for an almost paracontact metric manifold can be found $\{\vartheta_{1_1},\dots,\vartheta_{1_n},\vartheta_{2_1},\dots,\vartheta_{2_n},\zeta\}$, such that $g\left(\vartheta_{1_i},\vartheta_{1_j}\right)=-g\left(\vartheta_{2_i},\vartheta_{2_j}\right)=\delta_{ij}$ and $\vartheta_{2_i}=\phi\vartheta_{2_i}$, for any $i,j\in\{1,\dots,n\}$. Moreover, it is possible to establish the definition of a skew-symmetric tensor field (a 2-form), commonly referred to as the fundamental form, denoted as Φ, by using the equation

$$\Phi(\vartheta_1,\vartheta_2)=g(\vartheta_1,\phi\vartheta_2).$$

Definition 1:[5] If

$$(\nabla_{\vartheta_1}\phi)\vartheta_2=\alpha[-g(\vartheta_1,\vartheta_2)\zeta+\eta(\vartheta_2)\vartheta_1]+\beta[g(\vartheta_1,\phi\vartheta_2)\zeta+\eta(\vartheta_2)\phi\vartheta_1] \quad (4)$$

then the manifold $(M^{2n+1},\phi,\eta,\zeta,g)$ is called a tpS manifold.

A $(2n+1)$-dimensional tpS manifold satisfies followings [5]:

$$\nabla_{\vartheta_1}\zeta=-\alpha\phi\vartheta_1-\beta(\vartheta_1-\eta(\vartheta_1)\zeta), \tag{5}$$

$$(\nabla_{\vartheta_1}\eta)\vartheta_2=\alpha g(\vartheta_1,\phi\vartheta_2)-\beta\big(g(\vartheta_1,\vartheta_2)-\eta(\vartheta_1)\eta(\vartheta_2)\big), \tag{6}$$

$$\begin{aligned} R(\vartheta_1,\vartheta_2)\zeta= & -(\alpha^2+\beta^2)(\eta(\vartheta_2)\vartheta_1-\eta(\vartheta_1)\vartheta_2)-2\alpha\beta(\eta(\vartheta_2)\phi\vartheta_1-\eta(\vartheta_1)\phi\vartheta_2)\\ & -\vartheta_1(\alpha)\phi\vartheta_2+\vartheta_2(\alpha)\phi\vartheta_1+\vartheta_2(\beta)\phi^2\vartheta_1-\vartheta_1(\beta)\phi^2\vartheta_2, \end{aligned} \tag{7}$$

$$\begin{aligned} \eta(R(\vartheta_1,\vartheta_2)\vartheta_3)= & (\alpha^2+\beta^2)[\eta(\vartheta_2)g(\vartheta_1,\vartheta_3)-\eta(\vartheta_1)g(\vartheta_2,\vartheta_3)]+2\alpha\beta\begin{bmatrix}\eta(\vartheta_2)g(\phi\vartheta_1,\vartheta_3)\\ -\eta(\vartheta_1)g(\phi\vartheta_2,\vartheta_3)\end{bmatrix}\\ & +\vartheta_1(\alpha)g(\phi\vartheta_2,\vartheta_3)-\vartheta_2(\alpha)g(\phi\vartheta_1,\vartheta_3)-\vartheta_2(\beta)g(\phi^2\vartheta_1,\vartheta_3)+\vartheta_1(\beta)g(\phi^2\vartheta_2,\vartheta_3), \end{aligned} \tag{8}$$

$$R(\zeta,\vartheta_1)\zeta=(\alpha^2+\beta^2-\zeta(\beta))(\vartheta_1-\eta(\vartheta_1)\zeta), \tag{9}$$

$$S(\vartheta_1,\zeta)=-\big(2n(\alpha^2+\beta^2)-\zeta(\beta)\big)\eta(\vartheta_1)+(2n-1)\vartheta_1(\beta)-\phi\vartheta_1(\alpha), \tag{10}$$

$$S(\zeta,\zeta)=-2n\big(\alpha^2+\beta^2-\zeta(\beta)\big), \tag{11}$$

$$2\alpha\beta-\zeta(\alpha)=0, \tag{12}$$

$$Q\zeta=-(2n(\alpha^2+\beta^2)-\zeta(\beta))\zeta+(2n-1)\mathrm{grad}\,\beta+\phi(\mathrm{grad}\,\alpha), \tag{13}$$

where R is the Riemannian curvature tensor, S is the Ricci tensor and Q is the Ricci operator defined by $S(\vartheta_1,\vartheta_2)=g(Q\vartheta_1,\vartheta_2)$. In [6], the authors give the following expressions for three-dimensional tpS manifolds:

$$\begin{aligned} Q\vartheta_1= & \left[\frac{r}{2}-\zeta(\beta)+(\alpha^2+\beta^2)\right]\vartheta_1-\left[\frac{r}{2}-\zeta(\beta)+3(\alpha^2+\beta^2)\right]\eta(\vartheta_1)\zeta\\ & +[\phi(\mathrm{grad}\,\alpha)+\mathrm{grad}\,\beta]\eta(\vartheta_1)+[\vartheta_1(\beta)-\phi\vartheta_1(\alpha)]\zeta, \end{aligned} \tag{14}$$

$$\begin{aligned} S(X,Y)= & \left[\frac{r}{2}-\zeta(\beta)+(\alpha^2+\beta^2)\right]g(\vartheta_1,\vartheta_2)-\left[\frac{r}{2}-\zeta(\beta)+3(\alpha^2+\beta^2)\right]\eta(\vartheta_1)\eta(\vartheta_2)\\ & +[\vartheta_2(\beta)-\phi\vartheta_2(\alpha)]\eta(\vartheta_1)+[\vartheta_1(\beta)-\phi\vartheta_1(\alpha)]\eta(\vartheta_2), \end{aligned} \tag{15}$$

$$\begin{aligned} R(\vartheta_1,\vartheta_2)\vartheta_3= & \left[\frac{r}{2}-2\zeta(\beta)+2(\alpha^2+\beta^2)\right](g(\vartheta_2,\vartheta_3)\vartheta_1-g(\vartheta_1,\vartheta_3)\vartheta_2)\\ & -g(\vartheta_2,\vartheta_3)\left(\left[\frac{r}{2}-\zeta(\beta)+3(\alpha^2+\beta^2)\right]\eta(\vartheta_1)\zeta-[\phi(\mathrm{grad}\,\alpha)+\mathrm{grad}\,\beta]\eta(\vartheta_1)-[\vartheta_1(\beta)-\phi\vartheta_1(\alpha)]\zeta\right)\\ & +g(\vartheta_1,\vartheta_3)\left(\left[\frac{r}{2}-\zeta(\beta)+3(\alpha^2+\beta^2)\right]\eta(\vartheta_2)\zeta-[\phi(\mathrm{grad}\,\alpha)+\mathrm{grad}\,\beta]\eta(\vartheta_2)-[\vartheta_2(\beta)-\phi\vartheta_2(\alpha)]\zeta\right)\\ & -\left(\left[\frac{r}{2}-\zeta(\beta)+3(\alpha^2+\beta^2)\right]\eta(\vartheta_2)\eta(\vartheta_3)-[\vartheta_3(\beta)-\phi\vartheta_3(\alpha)]\eta(\vartheta_2)-[\vartheta_2(\beta)-\phi\vartheta_2(\alpha)]\eta(\vartheta_3)\right)\vartheta_1\\ & +\left(\left[\frac{r}{2}-\zeta(\beta)+3(\alpha^2+\beta^2)\right]\eta(\vartheta_1)\eta(\vartheta_3)-[\vartheta_3(\beta)-\phi\vartheta_3(\alpha)]\eta(\vartheta_1)-[\vartheta_1(\beta)-\phi\vartheta_1(\alpha)]\eta(\vartheta_3)\right)\vartheta_2. \end{aligned} \tag{16}$$

3. Results And Discussion

3.1. 3-Dimensional tpS Manifold Admitting η-Parallel Ricci Tensor

Definition 2: [8] The Ricci tensor S of a tpS manifold is called *η-parallel* if the following holds

$$(\nabla_{\vartheta_1} S)(\phi\vartheta_2, \phi\vartheta_3) = 0, \tag{17}$$

for all $\vartheta_1, \vartheta_2, \vartheta_3 \in \chi(M)$.

Theorem 1: *A 3-dimensional tpS manifold has η-parallel Ricci tensor for α and β are constants iff the manifold has constant scalar curvature.*

Proof: Letting $\vartheta_1 = \phi\vartheta_1$ and $\vartheta_2 = \phi\vartheta_2$ in (15), we get

$$S(\phi\vartheta_1, \phi\vartheta_2) = \left[\frac{r}{2} - \zeta(\beta) + (\alpha^2 + \beta^2)\right]\left(-g(\vartheta_1, \vartheta_2) + \eta(\vartheta_1)\eta(\vartheta_2)\right). \tag{18}$$

After differentiating (18) covariantly and using (4), (5) and (10), we obtain

$$\begin{aligned}(\nabla_{\vartheta_3} S)(\phi\vartheta_1, \phi\vartheta_2) = &\left[\frac{dr(\vartheta_3)}{2} - \nabla_{\vartheta_3}\zeta(\beta) + 2\alpha d\alpha(\vartheta_3) + 2\beta d\beta(\vartheta_3)\right]\left(-g(\vartheta_1, \vartheta_2) + \eta(\vartheta_1)\eta(\vartheta_2)\right)\\ &+\left(\phi\vartheta_2(\beta) - \phi^2\vartheta_2(\alpha)\right)\left[\alpha\left(g(\vartheta_1, \vartheta_3) - \eta(\vartheta_1)\eta(\vartheta_3)\right) - \beta g(\vartheta_3, \phi\vartheta_1)\right]\\ &+(\phi\vartheta_1(\beta) - \phi^2\vartheta_1(\alpha))[\alpha(g(\vartheta_2, \vartheta_3) - \eta(\vartheta_2)\eta(\vartheta_3)) - \beta g(\vartheta_3, \phi\vartheta_2)].\end{aligned} \tag{19}$$

Suppose that α and β are constants. Then, from (19) we get

$$(\nabla_{\vartheta_3} S)(\phi\vartheta_1, \phi\vartheta_2) = \frac{dr(\vartheta_3)}{2}\left(-g(\vartheta_1, \vartheta_2) + \eta(\vartheta_1)\eta(\vartheta_2)\right). \tag{20}$$

Let $\{v_i\}$ be a local orthonormal basis. Putting $\vartheta_1 = \vartheta_2 = v_i$ in (20) and taking the summation over i, we derive

$$0 = \frac{dr(\vartheta_3)}{2}\sum_{i=1}^{3} \varepsilon_i\left(-g(v_i, v_i) + \eta(v_i)\eta(v_i)\right), \tag{21}$$

which implies $dr(\vartheta_3) = 0$, i.e. r is constant.

3.2. 3-Dimensional tpS Manifold Admitting Cyclic Parallel Ricci Tensor

We will consider 3-dimensional tpS manifolds which admits cyclic parallel Ricci tensor.

Definition 3: [9] A semi-Riemannian manifold is said to admit cyclic parallel Ricci tensor if

$$(\nabla_{\vartheta_1} S)(\vartheta_2, \vartheta_3) + (\nabla_{\vartheta_2} S)(\vartheta_3, \vartheta_1) + (\nabla_{\vartheta_3} S)(\vartheta_1, \vartheta_2) = 0, \tag{22}$$

for all $\vartheta_1, \vartheta_2, \vartheta_3 \in \chi(M)$.

From (16), we have the following result.

Theorem 2: *If a 3-dimensional tpS manifold for α and β are constants has constant scalar curvature r, then $r = -6(\alpha^2 + \beta^2)$.*

Theorem 3: *A 3-dimensional tpS manifold for α and β are constants which is not α-para-Sasakian manifold satisfies cyclic parallel Ricci tensor iff the scalar curvature $r = -6(\alpha^2 + \beta^2)$.*

Proof: Since α and β are constants, from (15), we have

$$S(\vartheta_1,\vartheta_2)=\left(\frac{r}{2}+(\alpha^2+\beta^2)\right)g(\vartheta_1,\vartheta_2)-\left(\frac{r}{2}+3(\alpha^2+\beta^2)\right)\eta(\vartheta_1)\eta(\vartheta_2). \tag{23}$$

We also know that, $(\nabla_{\vartheta_1}S)(\vartheta_2,\vartheta_3)=\nabla_{\vartheta_1}S(\vartheta_2,\vartheta_3)-S(\nabla_{\vartheta_1}\vartheta_2,\vartheta_3)-S(\vartheta_2,\nabla_{\vartheta_1}\vartheta_3)$. Using (23) in the above equation, we obtain

$$(\nabla_{\vartheta_1}S)(\vartheta_2,\vartheta_3)=-\left(\frac{r}{2}+3(\alpha^2+\beta^2)\right)\left(g(\nabla_{\vartheta_1}\zeta,\vartheta_2)\eta(\vartheta_3)+g(\nabla_{\vartheta_1}\zeta,\vartheta_3)\eta(\vartheta_2)\right). \tag{24}$$

Let $\{v,\phi v,\zeta\}$ be a local orthonormal basis. Using (5) and (24) in (22) and taking the summation with respect to a local orthonormal basis, we get

$$\beta\left(\frac{r}{2}+3(\alpha^2+\beta^2)\right)\eta(\vartheta_1)=0. \tag{25}$$

which it gives $r=-6(\alpha^2+\beta^2)$.

Example. Let M be a three-dimensional manifold and the vector fields

$$\vartheta_1=\frac{\partial}{\partial u},\ \vartheta_2=\frac{\partial}{\partial v},\ \vartheta_3=(u+v)\frac{\partial}{\partial u}+(u+v)\frac{\partial}{\partial v}+\frac{\partial}{\partial w},$$

where

$$g=\begin{pmatrix}1 & 0 & -\frac{u+v}{2}\\ 0 & -1 & \frac{u+v}{2}\\ -\frac{u+v}{2} & \frac{u+v}{2} & 1\end{pmatrix},$$

$$\phi=\begin{pmatrix}0 & 1 & -(u+v)\\ 1 & 0 & -(u+v)\\ 0 & 0 & 0\end{pmatrix}.$$

One can observe that,

$$g(\vartheta_1,\vartheta_1)=g(\vartheta_3,\vartheta_3)=1, g(\vartheta_2,\vartheta_2)=-1, g(\vartheta_1,\vartheta_2)=g(\vartheta_1,\vartheta_3)=g(\vartheta_2,\vartheta_3)=0$$

and

$$\phi(\vartheta_1)=\vartheta_2,\ \phi(\vartheta_2)=\vartheta_1,\ \phi(\vartheta_3)=0.$$

We get

$$[\vartheta_1,\vartheta_3]=\vartheta_1+\vartheta_2,\ [\vartheta_2,\vartheta_3]=\vartheta_1+\vartheta_2,\ [\vartheta_1,\vartheta_2]=0,$$

Taking $\vartheta_3=\zeta$ and using Koszul formula, we can calculate

$$\nabla_{\vartheta_1}\vartheta_1 = -\zeta, \quad \nabla_{\vartheta_2}\vartheta_1 = 0, \quad \nabla_{\vartheta_3}\vartheta_1 = -\vartheta_2$$
$$\nabla_{\vartheta_1}\vartheta_2 = 0, \quad \nabla_{\vartheta_2}\vartheta_2 = \zeta, \quad \nabla_{\vartheta_3}\vartheta_2 = -\vartheta_1$$
$$\nabla_{\vartheta_1}\vartheta_3 = \vartheta_1, \quad \nabla_{\vartheta_2}\vartheta_3 = \vartheta_2, \quad \nabla_{\vartheta_3}\vartheta_3 = 0.$$

We also see that

$$\begin{aligned}
(\nabla_{\vartheta_1}\phi)\vartheta_1 &= \nabla_{\vartheta_1}\phi(\vartheta_1) - \phi(\nabla_{\vartheta_1}\vartheta_1) = -0 \\
&= 0(-g(\vartheta_1,\vartheta_1)\zeta + \eta(\vartheta_1)\vartheta_1) - 1\left(g(\vartheta_1,\phi(\vartheta_1))\zeta + \eta(\vartheta_1)\phi(\vartheta_1)\right) \\
(\nabla_{\vartheta_1}\phi)\vartheta_2 &= \nabla_{\vartheta_1}\phi(\vartheta_2) - \phi(\nabla_{\vartheta_1}\vartheta_2) = -\zeta \\
&= 0(-g(\vartheta_1,\vartheta_2)\zeta + \eta(\vartheta_2)\vartheta_1) - 1\left(g(\vartheta_1,\phi(\vartheta_2))\xi + \eta(\vartheta_2)\phi(\vartheta_1)\right) \\
(\nabla_{\vartheta_1}\phi)\vartheta_3 &= \nabla_{\vartheta_1}\phi(\vartheta_3) - \phi(\nabla_{\vartheta_1}\vartheta_3) = -\vartheta_2 \\
&= 0(-g(\vartheta_1,\vartheta_3)\zeta + \eta(\vartheta_3)\vartheta_1) - 1\left(g(\vartheta_1,\phi(\vartheta_3))\zeta + \eta(\vartheta_3)\phi(\vartheta_1)\right).
\end{aligned}$$

In the above equations, we see that the manifold satisfies the condition (4) for $X = \vartheta_1, \alpha = 0, \beta = -1$ and $\vartheta_3 = \zeta$. Similarly, it is also true for $X = \vartheta_2$ and $X = \vartheta_3$. The 1-form $\eta = dw$ and the fundamental 2-form $\Phi = du \wedge dv - (u + v)du \wedge dw + (u + v)dv \wedge dw$ defines a tpS manifold, where $d\eta = \alpha\Phi$, $d\Phi = -2\beta\eta \wedge \Phi$. So, the manifold is a tpS manifold for $\alpha = 0, \beta = -1$.

Then the expressions of the curvature tensor is given by

$$\begin{aligned}
&R(\vartheta_1,\vartheta_2)\vartheta_3 = 0, \quad R(\vartheta_2,\vartheta_3)\vartheta_3 = -\vartheta_2, \quad R(\vartheta_1,\vartheta_3)\vartheta_3 = -\vartheta_1, \\
&R(\vartheta_1,\vartheta_2)\vartheta_2 = \vartheta_1, \quad R(\vartheta_2,\vartheta_3)\vartheta_{2=} - \zeta, \quad R(\vartheta_1,\vartheta_3)\vartheta_2 = 0, \\
&R(\vartheta_1,\vartheta_2)\vartheta_1 = \vartheta_2, \quad R(\vartheta_2,\vartheta_3)\vartheta_1 = 0, \quad R(\vartheta_1,\vartheta_3)\vartheta_1 = \zeta.
\end{aligned} \tag{26}$$

Therefore, we have $S(\vartheta_1,\vartheta_1) = -2, S(\vartheta_2,\vartheta_2) = 2$ and $S(\vartheta_3,\vartheta_3) = -2$. It implies that the scalar curvature $r = -6$. Also using (15), (23) and (26), we can see that this example satisfy Theorem 1 and Theorem 3.

4. Conclusion

In the current paper, we examine a three-dimensional tpS manifold under some special conditions. First, we study a three-dimensional tpS manifold for α and β are constants which admit η-parallel Ricci tensor and show that the manifolds have constant scalar curvature. Then, we compute the scalar curvature of a three-dimensional tpS manifold which admit cyclic parallel Ricci tensor. At the end, we provide an example that supports our results. We think this paper is interesting and it will shed light on new studies about tpS manifolds.

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