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Research Article

On a non-riemannian quantity of (α,β)-metrics

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ABSTRACT

In this paper, we study a non-Riemannian quantity χ-curvature of $(\alpha,\ \beta)$-metrics, a special class of Finsler metrics with Riemannian metric α and a 1-form β . We prove that every $(\alpha,\ \beta)$-metric has a vanishing χ-curvature under certain conditions.

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INTRODUCTION

M. Matsumoto defined (α,β)-metrics [1] as a generalization of the Randers metric. One can find applications of (α,β)-metrics in physics and biology [2-4] in 1972. We understand geometric properties of Finsler metrics in the general case better as we study (α,β)-metrics more. Although it is more difficult to study (α,β)-metrics, when compared to studying Randers metric, we see good results with full of geometric properties appearing for (α,β)-metrics in recent years, [5-10].

An (α,β)-metric is a scalar function on TM defined by $F=\alpha\emptyset(s)$, $s=\frac{\beta}{\alpha}$ where $\emptyset=\emptyset(s)$ is a C^{∞} function on $(-b_{0,}b_0)$, $\alpha=\sqrt{a_{ij}(x)y^i y^j}$ is a Riemannian metric and $\beta=\ b_i(x)y^i$ is a 1-form. It can be shown that for any Riemannian metric α and any 1-form β on M with $b=||\beta_x||_\alpha<\ b_0$ the function $F=\alpha\emptyset\left(\frac{\beta}{\alpha}\right)$ is a (positive definite) Finsler metric if and only if $\emptyset$ satisfies

$$\emptyset(s)>0,\ \ \emptyset(s)-s\emptyset^{'}(s)+(b^2-s^2)\emptyset^{''}(s)>0,$$
$$(|s|\leq\ \rho<b_0\ ).$$

Such (α, β)-metrics are said to be regular. Randers metrics are special (α, β)- metrics defined by $\emptyset = 1 + s$, i.e., F = α + β.

Non-Riemannian quantities play a quite important role in Finsler geometry [11-15]. They all vanish for Riemannian metrics [13, 16-17]. In this paper we consider a few non-Riemannian quantities. The χ-curvature, H- curvature, S-curvature are some of the non-Riemannian quantities in Finsler geometry. The Riemann curvature

$$R\ =\ {R^i}_k\frac{\partial}{\partial x^i}\otimes dx^k$$

is defined by

$${R^i}_k:=2\frac{\partial G^i}{\partial x^k}-y^j\frac{\partial^2G^i}{\partial x^j\partial y^k}+2G^j\frac{\partial^2G^i}{\partial y^j\partial y^k}-\frac{\partial G^i}{\partial y^j}\frac{\partial G^j}{\partial y^k}$$

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where $G^i = G^i(x, y)$ are the spray coefficients. The χ-curvature is related to the Riemannian curvature as shown below,

$$\chi_i = -\frac{1}{3}\{2R^m{}_{i\cdot m} + R^m{}_{m\cdot i}\}.$$

The χ-curvature, H-curvature, S-curvature are related to each other as shown below.

$$\chi_i = S_{.i|m}y^m - S_{|i},$$
$$H_{ij} = \frac{1}{4}\{\chi_{j\cdot i} + \chi_{i\cdot j}\},$$
$$H_{ij} = \frac{1}{2} S_{.i.j|m}y^m,$$

where $\chi := \chi_j\, dx^j$ and $H := H_{ij}dx^i \otimes dx^j$ denote the χ- and H -curvatures of F on the tangent bundle TM, respectively, S denotes the S -curvature of F. The notations " · " and " | " denote the vertical and horizontal covariant derivatives with respect to the Chern connection of F, respectively, [14]. The Chern connection solves the equivalence problem for Finsler structure, like many other connections, [18], and gives rise to a list of criteria to decide when two such structures differ only by a change of coordinates, [19].

Main Theorem and Some Applications

In this paper, we study a non-Riemannian quantity, the χ-curvature of (α, β)-metrics, a special and a large class of Finsler metrics with Riemannian metric α and a 1-form β. We prove that every (α, β)-metric has a vanishing χ-curvature under certain conditions. We give the following theorem and its corollary below.

Theorem 1.1 Let F be an (α, β)-metric on n-dimensional manifold M. Assume that β is a closed 1-form and $r_{ij} = K(b^2a_{ij} - b_ib_i)$, where K is a constant. Then the χ-curvature vanishes.

One can see that the following corollary can easily be proven when we apply the given conditions to the equation in (3.27).

Corollary 1.2 Let F be an (α, β)-metrics on n-dimensional manifold M. β is a closed 1-form and $r_{ij} = 0$ if and only if the χ-curvature vanishes.

As an application we give two examples using a Randers metric defined on S^3 and an (α, β)-metric defined on an open subset in R^3, respectively. These examples show that under the given conditions, namely, when β is a closed 1-form and $r_{ij} = K(b^2a_{ij} - b_ib_i)$, where K is a constant, we get that the χ- curvature vanishes.

Randers metrics were first introduced by the physicist G. Randers in 1941 in the theory of general relativity where R. S. Ingarden, was the one who named it as Randers Metrics for the first time in an application in his thesis [20] in the theory of the electron microscope. They are among the simplest Finsler metrics, expressed in the form F = α + β, where $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ is a Riemannian metric on a differentiable manifold M and $\beta = b_i(x)y^i$ is a 1-form with $||\beta_x||_\alpha := sup_{y\in T_xM} \frac{|\beta(y)|}{\alpha(y)} < 1$ for any point $x \in M$. Many Finsler geometers have studied the geometric properties of Randers metrics and have obtained many important and interesting results, [21-26].

Example 1 [23] Let F = α + β be the family of Randers metrics on S^3 constructed in [27]. It is shown that $r_{ij} = 0$ and $s_j = 0$. Thus for any C^∞ positive function $\varphi = \varphi(s)$ satisfying the following

$$\varphi(s) - s\varphi'(s) + (\rho^2 - s^2)\varphi''(s) > 0, \ (|s| \leq \rho < b_0),$$

the (α, β)-metric $F = \alpha\varphi(\frac{\beta}{\alpha})$ has vanishing S -curvature. This implies that the χ-curvature vanishes.

Example 2 [23] Let $F = \alpha\varphi(\frac{\beta}{\alpha})$ be an (α, β)-metric defined on an open subset in $\mathbb{R}^3$. At a point

x = (x, y, z) in $\mathbb{R}^3$ and in the direction y = (u, v, w) in $T_x\mathbb{R}^3$, α = α(x, y) and β = β(x, y) are given by

$\alpha := \sqrt{u^2 + e^{2x}(v^2 + w^2)}$, β: = u. Then β satisfies $r_{ij} = \varepsilon(b^2a_{ij} - b_ib_j)$, $s_j = 0$ with $\varepsilon = 1,\ b = 1$.

Thus if $\varphi = \varphi(s)$ satisfies

$$\phi = -2(n+1)k\frac{\varphi\Delta^2}{(b^2 - s^2)},$$

for some constant k, then $F = \alpha\varphi(\frac{\beta}{\alpha})$ is of constant S -curvature, namely S = (n + 1), hence (1.3) implies that the χ-curvature vanishes.

Proof of the main Theorem

In this section we prove the main Theorem and the given Proposition below. Before the proofs, we introduce some facts first. Let F be an (α,β)-metric on a manifold M^n defined in the previous section and the spray coefficients G^i of F has the following formula given below

$$G^i = \bar{G}^i + H^i,$$

where $\bar{G}^i$ denote the spray coefficients of α and

$$H^i = \alpha Q s^i_0 + \{-2Q\alpha s_0 + r_{00}\}\{\theta\frac{y^i}{\alpha} + \Psi b^i\} \qquad (1)$$

with

$$Q = \frac{\varphi'}{\varphi - s\varphi'};\ \Delta = 1 + sQ + (b^2 - s^2)Q';\ \psi = \frac{Q'}{2\Delta};\ \Theta = \frac{Q - sQ'}{2\Delta}; \qquad (2)$$

$b = ||\beta_x||_\alpha$, $s^i_j = a^{ik}s_{kj}$, $s^i_0 = s^i_j\, y^j$, $s_0 = s_i\, y^i$, $r_{00} = r_{ij}\, y^i y^j$.

Note that the index '0' means 'contracting with respect to y'. We also express

$$\Pi = \bar{\Pi} + \Gamma,$$

where $\Pi = [G^m]_{y^m}$, $\bar{\Pi} = [\bar{G}^m]_{y^m}$, and $\Gamma = [H^m]_{y^m}$. Using the above identities, we get

$$\Gamma = 2\Xi s_0 + 2r_0\Psi - \alpha^{-1}\phi r_{00}, \qquad (3)$$

where

$$\Xi = \Psi + \bar{\phi}\, Q \tag{4}$$

$$\bar{\Phi} = \frac{1}{2\Delta^2}\{-(Q - sQ')(n\Delta + 1 + sQ) - (b^2 - s^2)(1 + sQ)Q''\} \tag{5}$$

The χ-curvature is given by

$$X_i = \frac{1}{2}\{\Pi_{y^i x^m}\, y^m - \Pi_{x^i} - 2\Pi_{y^i y^m} G^m\}, \tag{6}$$

where Π and G^m are as defined above. In the next Proposition 3.1, we compute χ_i by using (6).

Proposition 3.1 Let F be an (α, β)-metrics on n-dimensional manifold M. Then the χ-curvature is given below,

$$X_i = C + D + E \tag{7}$$

where

$$\begin{aligned}
C &= \alpha^{-1}(2\Xi_s s_i + 2r_i\Psi s - \alpha^{-1} r_{i0}\,\bar{\Phi}s)\, r_{00} \\
&+ 2\alpha^{-1}(2\Xi_s s_0 + 2r_0\Psi s - \alpha^{-1} r_{00}\,\bar{\Phi}s)\, s_{i0} \\
&+ \alpha^{-2}(2\Xi_s s_{0;0} + 2r_{0;0}\Psi s - \alpha^{-1} r_{00;0}\,\bar{\Phi}s)\, h_I \\
&+ 2\alpha^{-2}(2\Xi_{s\rho} s_0 + 2r_0\Psi s\rho - \alpha^{-1} r_{00}\,\bar{\Phi}s\rho)(r_0 + s_0)\, h_i \\
&- 2\alpha^{-2}(2\Xi_s s_0 + 2r_0\Psi s - \alpha^{-1} r_{00}\,\bar{\Phi}s)\, h_{i.m} \\
&H^m - 2\alpha^{-2}(2\Xi_s s_i + 2r_i\Psi s - \alpha^{-1} r_{i0}\,\bar{\Phi}s)\, h_m H^m \\
&- 2\alpha^{-2}(2\Xi_s s_m + 2r_m\Psi s - \alpha^{-1} r_{m0}\,\bar{\Phi}s)\, h_I \\
&H^m - \alpha^{-3}(2\Xi_s s_0 + 2r_0\Psi s - \alpha^{-1} r_{00}\,\bar{\Phi}s)\, r_{000}\, y_i,
\end{aligned}$$

$$\begin{aligned}
D &= \alpha^{-3}(2\Xi_{ss} s_0 + 2r_0\Psi ss - \alpha^{-1} r_{00}\,\bar{\Phi}ss) r_{000}\, h_i \\
&+ \alpha^{-3}\,\bar{\Phi}(r_{00;0}\, y_i - \alpha^2\, r_{i0;0}) \\
&+ 2\,\alpha^{-3}\,\bar{\Phi}\rho\,(r_0 + s_0)(r_{00}\, y_I - \alpha^2 r_{i0}) \\
&- 2\,\alpha^{-3}\,\bar{\Phi}\,(r_{00} a_{im} - \alpha^2\, r_{im})\, H^m \\
&- 4\,\alpha^{-3}\,\bar{\Phi}\,(r_{m0}\, y_i - \alpha^2\, r_{im})\, H^m \\
&- 2\,\alpha^{-4}(2\Xi_{ss} s_0 + 2r_0\Psi ss - \alpha^{-1} r_{00}\,\bar{\Phi}ss) h_i\, h_m H^m \\
&+ 4\,\alpha^{-4}(2\Xi_s s_0 + 2r_0\Psi s - \alpha^{-1} r_{00}\,\bar{\Phi}s)\, h_i\, y_m\, H^m \\
&+ \alpha^{-4}\,\bar{\Phi}s\, r_{00}(r_{00}\, y_i - \alpha^2\, r_{i0})
\end{aligned}$$

$$\begin{aligned}
E &= 2\,\alpha^{-5}\,\bar{\Phi}s\,(r_{00}\, y_i - \alpha^2\, r_{i0})\, h_m\, H^m \\
&- 2\,\alpha^{-5}\,\bar{\Phi}s\,(r_{00}\, y_m - \alpha^2\, r_{m0})\, h_i\, H^m \\
&+ 6\,\alpha^{-5}\,\bar{\Phi}\,(r_{00} y_i - \alpha^2\, r_{i0})\, y_m\, H^m \\
&- 2\,(2\Xi_\rho s_0 + 2r_0\Psi\rho - \alpha^{-1} r_{00}\,\bar{\Phi}\rho)\,(r_i + s_i) \\
&+ 2\,(2\Xi_\rho s_I + 2r_i\Psi\rho - \alpha^{-1} r_{i0}\,\bar{\Phi}\rho)(r_0 + s_0) \\
&+ (2\Xi_{s_{i;0}} s_i + 2r_{i;0}\Psi - \alpha^{-1} r_{i0;0}\,\bar{\Phi}) \\
&- (2\Xi_{s_{0;i}} s_i + 2r_{0;i}\Psi - \alpha^{-1} r_{00;i}\,\bar{\Phi}).
\end{aligned}$$

where H_m, Q, θ, y, D, χ and $\bar{\Phi}$ are given above in (1), (2), and (5), respectively.

Proof of the Proposition

We recall Γ defined in (3), and we calculate the following terms $\Gamma_{;i}$, $\Gamma_{.i}$, $\Gamma_{.i;m}y^m$, $\Gamma_{.i.m}H^m$ below. When we let $\rho := b^2$, we get $\rho_{;m}y^m = 2(r_0 + s_0)$ We first need the following useful identities for $h_i = \alpha b_i - sy_i$ below in (8),

$$\begin{aligned}
&s_{.i} = \alpha^{-2} h_i\;;\; s_{;m} = \alpha^{-1}(r_{m0} - s_{m0});\; \alpha_{.i} = \alpha^{-1} y_i\;;\; \alpha_{;i} = 0; \\
&h_{i;m} y^m = \alpha\,(r_{i0} - s_{i0}) - \alpha^{-1} r_{00}\, y_i\,; \\
&s_{;m}\, y^m = \alpha^{-1} r_{00}\,;\; y_{i;m}\, y^m = 0;
\end{aligned} \tag{8}$$

$$\begin{aligned}
\Gamma_{;i} &= \alpha^{-1}(2\Xi_s s_0 + 2r_0\Psi s - \alpha^{-1} r_{00}\,\bar{\Phi}s)(r_{i0} - s_{i0}) \\
&+ 2(2\Xi_\rho s_0 + 2r_0\Psi\rho - \alpha^{-1} r_{00}\,\bar{\Phi}\rho)(r_i + s_i) \\
&+ (2\Xi s_{0;i} + 2r_{0;i}\Psi - \alpha^{-1} r_{00;i}\,\bar{\Phi})
\end{aligned} \tag{9}$$

$$\begin{aligned}
\Gamma_{.i} &= \alpha^{-2}(2\Xi_s s_0 + 2r_0\Psi s - \alpha^{-1} r_{00}\,\bar{\Phi}s)\, h_i \\
&+ (2\Xi s_i + 2r_i\Psi - \alpha^{-1} r_{i0}\,\bar{\Phi}) \\
&+ (2\Xi s_{0;i} + 2r_{0;i}\Psi - \alpha^{-1} r_{00;i}\,\bar{\Phi}) \\
&+ \alpha^{-3}\,\bar{\Phi}\,(r_{00}\, y_i - \alpha^2\, r_{i0}),
\end{aligned} \tag{10}$$

$$\Gamma_{.i;m}\, y^m = J + L, \tag{11}$$

where

$$\begin{aligned}
J &= \alpha^{-3} r_{00}(2\Xi_{ss} s_0 + 2r_0\Psi ss - \alpha^{-1} r_{00}\,\bar{\Phi}ss)\, h_i \\
&+ 2\,\alpha^{-2}(2\Xi_{s\rho} s_0 + 2r_0\Psi s\rho - \alpha^{-1} r_{00}\,\bar{\Phi}s\rho)(r_0 + s_0) h_i \\
&+ \alpha^{-2}(2\Xi_s s_{0;0} + 2r_{0;0}\Psi s - \alpha^{-1} r_{00;0}\,\bar{\Phi}s)\, h_i \\
&+ \alpha^{-1}(2\Xi_s s_0 + 2r_0\Psi s - \alpha^{-1} r_{00}\,\bar{\Phi}s)(r_{i0} - s_{i0}) \\
&- \alpha^{-3} r_{00}(2\Xi_s s_0 + 2r_0\Psi s - \alpha^{-1} r_{00}\,\bar{\Phi}s)\, y_i
\end{aligned}$$

and

$$\begin{aligned}
L &= \alpha^{-1} r_{00}\,(2\Xi_s s_i + 2r_i\Psi s - \alpha^{-1} r_{i0}\,\bar{\Phi}s) \\
&+ (2\Xi_{s_{i;0}} + 2r_{i;0}\Psi - \alpha^{-1} r_{i0;0}\,\bar{\Phi}) \\
&+ 2(2\Xi_\rho s_i + 2r_i\Psi\rho - \alpha^{-1} r_{i0}\,\bar{\Phi}\rho)(r_0 + s_0) \\
&+ \alpha^{-4}\,\bar{\Phi}s\, r_{00}(r_{00}\, y_i - \alpha^2\, r_{i0}) \\
&+ 2\,\alpha^{-3}\,\bar{\Phi}\rho\,(r_0 + s_0)(r_{00}\, y_i - \alpha^2\, r_{i0}) \\
&+ \alpha^{-3}\,\bar{\Phi}\,(r_{00;0}\, y_i - \alpha^2\, r_{i0;0}),
\end{aligned}$$

$$\Gamma_{.i;m}\, y^m - \Gamma_{;i} = M + N + P, \tag{12}$$

where

$$\begin{aligned}
M = {} & \alpha^{-3}\,(2\Xi_{ss}s_0 + 2r_0\Psi ss - \alpha^{-1}\, r_{00}\,\Phi ss)r_{00}\, h_i \\
& + 2\,\alpha^{-2}\,(r_0 + s_0)\big(2\Xi_{s\rho}s_0 + 2r_0\Psi s\rho - \alpha^{-1}\, r_{00}\,\bar{\Phi} s\rho\big)\, h_I \\
& + \alpha^{-2}\big(2\Xi_s s_{0;0} + 2r_{0;0}\Psi s - \alpha^{-1}\, r_{00;0}\,\bar{\Phi} s\big)\, h_i \\
& + \alpha^{-1}(2\Xi_s s_0 + 2r_0\Psi s - \alpha^{-1}\, r_{00}\,\bar{\Phi} s)\,(r_{i0} + s_{i0})
\end{aligned}$$

$$\begin{aligned}
N = {} & \alpha^{-3}\,(2\Xi_s s_0 + 2r_0\Psi s - \alpha^{-1}\, r_{00}\,\Phi s)r_{00}\, y_i \\
& + \alpha^{-1}(2\Xi_s s_i + 2r_i\Psi s - \alpha^{-1}\, r_{i0}\,\bar{\Phi} s\,)r_{00} \\
& + \big(2\Xi\, s_{i;0} + 2r_{i;0}\Psi - \alpha^{-1}\, r_{i0;0}\,\bar{\Phi}\big) \\
& + 2(r_0 + s_0)\big(\,2\Xi_\rho s_i + 2r_i\Psi\rho - \alpha^{-1} r_{i0}\,\bar{\Phi}\rho\,\big) \\
& - \alpha^{-1}(2\Xi_s s_0 + 2r_0\Psi s - \alpha^{-1}\, r_{00}\,\bar{\Phi} s)(s_{io} - r_{i0})
\end{aligned}$$

$$\begin{aligned}
P = {} & 2(r_i - s_i)\big(2\Xi_\rho s_0 + 2r_0\Psi\rho - \alpha^{-1} r_{00}\,\bar{\Phi}\rho\big) \\
& - \big(2\Xi\, s_{0;i} + 2r_{0;i}\Psi - \alpha^{-1}\, r_{00;i}\,\bar{\Phi}\big) \\
& + \alpha^{-4}\,\bar{\Phi} s\, r_{00}\,(r_{00}\, y_i - \alpha^2\, r_{i0}) \\
& + 2\,\alpha^{-3}(r_0 + s_0)\bar{\Phi}\rho\,(\,r_{00}\, y_i - \alpha^2\, r_{i0}) \\
& + \alpha^{-3}\,\bar{\Phi}\,\big(r_{00;0}\, y_i - \alpha^2\, r_{i0;0}\big)
\end{aligned}$$

$$\begin{aligned}
\Gamma_{.i.m}H^m = {} & -2\,\alpha^{-4}(2\Xi_s s_0 + 2r_0\Psi s - \alpha^{-1}\, r_{00}\,\bar{\Phi} s)\, h_i\, y_m H^m \\
& + \alpha^{-4}(2\Xi_{ss}s_0 + 2r_0\Psi ss - \alpha^{-1} r_{00}\,\bar{\Phi} ss)\, h_i\, h_m H^m \\
& + \alpha^{-2}(2\Xi_s s_m + 2r_m\Psi s - \alpha^{-1}\, r_{m0}\,\bar{\Phi} s)\, h_i H^m \\
& + \alpha^{-2}(2\Xi_s s_0 + 2r_0\Psi s - \alpha^{-1}\, r_{00}\,\bar{\Phi} s)\, h_{i.m}H^m \\
& + \alpha^{-2}(2\Xi_s s_i + 2r_i\Psi s - \alpha^{-1}\, r_{i0}\,\bar{\Phi} s)\, h_m H^m \\
& + \alpha^{-5}\,\bar{\Phi} s\,\big(r_{00}\, y_m - \alpha^2 r_{m0}\big)\, h_i H^m \\
& + \alpha^{-5}(r_{00}\, y_i - \alpha^2 r_{i0})H^m\,(h_m\bar{\Phi} s - 3\, y_m\Phi) \\
& + \alpha^{-3}\Phi H^m\,(r_{00}\, a_{im} - 2\alpha^2 r_{im} - r_{i0}\, y_m + 2r_{m0}\, y_i)
\end{aligned} \tag{13}$$

where

$$\begin{aligned}
y_m H^m &= \alpha(r_{00} - 2\alpha Q s_0\,)\,(\theta + s\Psi), \\
s_{.m}H^m &= Qs_0 + \alpha^{-1}(b^2 - s^2)\,(r_{00} - 2\alpha Q s_0)\,\Psi, \\
r_m H^m &= \alpha Q r_m\, s_0{}^m + (r_{00} - 2\alpha Q s_0\,)\,(\alpha^{-1}\, r_0\,\theta + r\Psi), \\
s_m H^m &= \alpha Q s_m\, s_0{}^m + \alpha^{-1}\, s_0\,(r_{00} - 2\alpha Q s_0)\,\theta, \\
r_{m0}H^m &= \alpha Q r_{m0}\, s_0{}^m + (r_{00} - 2\alpha Q s_0)\,(\alpha^{-1}\, r_{00}\,\theta + r_0\,\Psi), \\
r_{im}H^m &= \alpha Q r_{im}\, s_0{}^m + (r_{00} - 2\alpha Q s_0)\,(\alpha^{-1}\, r_{i0}\,\theta + r_i\,\Psi), \\
a_{im}H^m &= \alpha Q s_{i0} + (r_{00} - 2\alpha Q s_0)\,(\alpha^{-1}\, y_i\,\theta + b_i\,\Psi), \\
h_{i.m}H^m &= -Q\,(\alpha\, s s_{i0} + y_i\, s_0) + \alpha^{-1}(r_{00} - 2\alpha Q s_0)(\theta h_i - (b^2 - s^2)\,\Psi y_i),
\end{aligned} \tag{14}$$

Plugging the equations (9)-(13) into (6), we obtain (7).

Proof of Theorem 2.1. We assume that β is a closed 1-form and we have the following further assumptions

$$r_{ij} = K\,(b^2 a_{ij} - b_i\, b_j), \tag{15}$$

$$s_{ij} = 0. \tag{16}$$

We obtain the following terms.

$$\begin{aligned}
& r_{00} = K(b^2 - s^2)\,\alpha^2\,; \quad r_I = 0; \quad r_0 = 0; \\
& r_{i0} = K(y_I\, b^2 - b_I\,\alpha\, s) = K(b^2 - s^2)y_I - Ks\, h_i\,; \\
& r_{i0} - \alpha^{-2} r_{00}\, y_i = -K\, s\, h_i; \;\; r_{i0;0} = -K\,(r_{i0}s\alpha + b_i\, r_{00}); \\
& r_{00;i} = -2K\, r_{i0}s\alpha; \;\; r_{00;0} = -2K\, r_{00}s\alpha.
\end{aligned}$$

We let $r_0 = 0$, $r_i = 0$, $s_{ij} = 0$ in the expression of χ_i in (3.11), then χ_i becomes as follows.

$$\begin{aligned}
X_i = {} & -\alpha^{-4}\bar{\Phi} ss\, r_{00}{}^2\, h_i + 2\,\alpha^{-5}\,\bar{\Phi} ss\, r_{00}\, h_i\, h_m H^m \\
& + 2\alpha^{-4}\,\bar{\Phi} s\, r_{00}{}^2\, y_i - 6\,\alpha^{-5}\bar{\Phi} s\, r_{00}\, h_i\, y_m H^m \\
& + 2\alpha^{-3}\,\bar{\Phi} s\, r_{00}\, h_{i.m}H^m - 2\,\alpha^{-2}\,\bar{\Phi} s\, r_{00}\, r_{i0} \\
& + 4\alpha^{-3}\,\bar{\Phi} s\, r_{i0}\, h_m\, H^m + 4\alpha^{-3}\,\bar{\Phi} s\, r_{m0}\, h_i H^m \\
& - \alpha^{-3}\,\bar{\Phi} s\, r_{00;0}\, h_i - 2\alpha^{-1}\,\bar{\Phi}\, r_{i0;0} + \alpha^{-1}\,\bar{\Phi}\, r_{00;i} \\
& + \alpha^{-3}\,\bar{\Phi}\, r_{00;0}\, y_i - 4\alpha^{-3}\,\bar{\Phi}\, H^m y_m\, r_{i0} \\
& - 2\,\alpha^{-5}\,\bar{\Phi} s\, r_{00}\, y_i\, h_m H^m + 6\,\alpha^{-5}\,\bar{\Phi}\, r_{00}\, y_i\, y_m H^m \\
& - 2\,\alpha^{-3}\,\bar{\Phi}\, r_{00}\, a_{im}\, H^m - 4\,\alpha^{-3}\,\bar{\Phi}\, r_{m0}\, y_i\, H^m \\
& + 4\,\alpha^{-1}\,\bar{\Phi}\, r_{im}\, H^m
\end{aligned}$$

We compute the following terms we need, and express them in terms of Θ, Ψ and others.

$$\begin{aligned}
& y_m H^m = \alpha\, r_{00}\,(\theta + s\Psi), \\
& r_{m0}H^m = \alpha^{-1}\,\theta\, r_{00}{}^2 \\
& r_{im}H^m = \alpha^{-1}\, r_{i0}\, r_{00}\,\theta, \\
& a_{im}H^m = r_{00}\,(\alpha^{-1}\, y_i\,\theta + b_i\,\Psi), \\
& h_{i.m}H^m = \alpha^{-1}\, r_{00}\,(\theta h_i - (b^2 - s^2)\,\Psi y_i\,), \\
& h_m H^m = \Psi\, r_{00}\,\alpha\,(b^2 - s^2),
\end{aligned}$$

We rewrite χ_i by using the above calculations,

$$\begin{aligned}
X_i = {} & -\alpha^{-4}\{\bar{\Phi} ss\, r_{00}{}^2 - 2\,\bar{\Phi} ss\, r_{00}{}^2\Psi\,(b^2 - s^2) \\
& + 6s\,\bar{\Phi} s\, r_{00}{}^2\,\Psi + 2\,\bar{\Phi}\, r_{00}{}^2\Psi + \alpha\,\bar{\Phi} s\, r_{00;0}\}\, h_i \\
& + 2\alpha^{-4}\,\{\,\bar{\Phi} s - 2\,\bar{\Phi} s\,\Psi\,(b^2 - s^2) + 2s\,\bar{\Phi}\,\Psi\}\, A \\
& + \alpha^{-3}\,\bar{\Phi}\, B,
\end{aligned} \tag{17}$$

where

$$\begin{aligned}
& A = r_{00}\,(r_{00}\, y_i - \alpha^2\, r_{i0}); \\
& B = \big(r_{00;0}\, y_i + \alpha^2\, r_{00;i} - 2\alpha^2\, r_{i0;0}\big).
\end{aligned} \tag{18}$$

Moreover, when we use (15), we get the following simplified terms,

$$A = K\, s\,\alpha^2\, h_i\, r_{00}\,; \quad B = 2\, K\,\alpha\, h_i\, r_{00}. \tag{19}$$

Using (19) into (17), we obtain the following compact expression for χ_i.

$$\begin{aligned}
X_i = {} & -\alpha^{-2}h_i\, r_{00}\, K\,\{-\bar{\Phi} ss\,(b^2 - s^2) \\
& + 2\,\bar{\Phi} ss\,(b^2 - s^2)^2\,\Psi - 10\, s\,\bar{\Phi} s\,\Psi(b^2 - s^2) \\
& - 2\bar{\Phi}\,\Psi\,(b^2 - 3s^2) + 4s\,\bar{\Phi} s + 2\,\bar{\Phi}\}
\end{aligned} \tag{20}$$

It is clear that $\bar{\Phi}$ in (5), could be rewritten as follows

$$\bar{\Phi} = \Phi/2\Delta^2, \text{ where } \bar{\Phi} = -(Q - sQ')(n\Delta + 1 + sQ) - (b^2 - s^2)(1 + sQ)Q''$$

We have the following fact that

$$\Phi = 2(n+1)\,k\,\varphi\,\Delta^2/(b^2 - s^2),$$
$$\Phi = (n+1)\,k\,\varphi\,/(b^2 - s^2),$$

We have the following derivatives for $\bar{\Phi}$.

$$\bar{\Phi} = (n+1)k\,\frac{\varphi}{(b^2-s^2)},$$

$$\bar{\Phi}s = (n+1)k\,\frac{\varphi_s}{(b^2-s^2)} - 2(n+1)k\,\frac{s\varphi}{(b^2-s^2)^2}, \tag{21}$$

$$\bar{\Phi}ss = -2(n+1)k\,\frac{\varphi}{(b^2-s^2)^2} - 8(n+1)k\,\frac{s^2\varphi}{(b^2-s^2)^3},$$

If we use $\bar{\Phi}$ in (5), we obtain

$$X_{i} = r_{00}\,K\,\alpha^{-2}h_i\,\{\varphi_{ss} - 2\,\varphi_{ss}\,(b^2 - s^2)\Psi - 2\,\varphi\,\Psi + 2s\,\varphi_s\,\Psi\}\,(n+1)k \tag{22}$$

Hence, using the equations below, we get $\chi_i = 0$

$$Q = \frac{\varphi'}{\varphi - s\varphi'};\quad \Delta = 1 + sQ + (b^2 - s^2)Q';$$
$$\psi = \frac{Q'}{2\Delta};\ \Theta = \frac{Q - sQ'}{2\Delta}.$$

CONCLUSION

Although this paper has a useful result expressing how a non-Riemannian quantity χ-curvature vanishes for (α, β)-metrics with Riemannian metric α and a 1-form β when β is a closed 1-form and $r_{ij} = K(b^2 a_{ij} - b_i b_i)$, where K is a constant as it is stated in the main theorem. We believe one can do more. One may consider doing computations to see what happens to the non-Riemannian quantity χ-curvature when K is a scalar function. This would allow us to see and explore the geometrical variations for β 1-form with respect to the Riemannian metric α. Next study will basically focus on this open problem and we hope to get a good result after rigorous and somewhat harder and more time consuming computations to be done on Maple.

AUTHORSHIP CONTRIBUTIONS

Authors equally contributed to this work.

DATA AVAILABILITY STATEMENT

The authors confirm that the data that supports the findings of this study are available within the article. Raw data that support the finding of this study are available from the corresponding author, upon reasonable request.

CONFLICT OF INTEREST

The author declared no potential conflicts of interest with respect to the research, authorship, and/or publication of this article.

ETHICS

There are no ethical issues with the publication of this manuscript.

REFERENCES

[1] Matsumoto M. On C-reducible Finsler spaces. Tensor NS 1972;24:29–37.

[2] Antonelli PL, Miron R. Lagrange and Finsler Geometry: Applications to Physics and Biology. Dordrecht: Kluwer Academic Publishers; 1996. (Fundamental Theories of Physics; vol 76). [CrossRef]

[3] Balan V, Stavrinos PC. Finslerian (α, β)-metrics in Weak Gravitational Models. In: Anastasiei M, Antonelli PL, editors. Finsler and Lagrange Geometries. Dordrecht: Kluwer Academic Publishers; 2003. [CrossRef]

[4] Roman M, Shimada H, Sabau VS. On β-change of the Antonelli Shimada ecological metric. Tensor NS 2004;65:65-73.

[5] Bao D, Chern SS, Shen Z. An Introduction to Riemann-Finsler Geometry. New York: Springer; 2000. [CrossRef]

[6] Kikuchi S. On the condition that a space with (α, β)-metric be locally Minkowskian. Tensor NS 1979;33:242–246.

[7] Matsumoto M. Finsler spaces with (α, β)-metric of Douglas type. Tensor NS 1998;60:123–134.

[8] Matsumoto M. The Berwald connection of a Finsler space with an (α, β)- metric. Tensor NS 1991;50:18–21.

[9] Sabau VS, Shimada H. Classes of Finsler spaces with (α, β)-metrics. Rep Math Phy 2001;47:31–48. [CrossRef]

[10] Shen Z. On a class of landsberg metrics in finsler geometry. Can J Math 2009;61:1357–1374. [CrossRef]

[11] Gabrani M, Rezaei B, Sevim ES. A Class of Finsler Metrics with Almost Vanishing H- and Ξ-curvatures. Results Math 2021;76:1–17. [CrossRef]

[12] Gabrani M, Rezai B, Sengelen Sevim E. General spherically symmetric Finsler metrics with constant Ricci and flag curvature. Differ Geom Appl 2021;76:1–19. [CrossRef]

[13] Najafi B, Shen Z, Tayebi A. Finsler metrics of scalar flag curvature with special non-Riemannian curvature properties. Geom Dedicata 2008;131:87–97. [CrossRef]

[14] Shen Z. On Some Non-Riemannian Quantities in Finsler Geometry. Can Math Bull 2013;56:184–193. [CrossRef]
[15] Tang D. On the non-Riemannian quantity H in Finsler geometry. Differ Geom Appl 2011;29:207–213. [CrossRef]
[16] Pişcoran LI, Mishra VN. S-curvature for a new class of (α, β)-metrics. Rev R Acad Cienc Exactas Fís Nat Ser A Mat 2017;111:1187–1200. [CrossRef]
[17] Najafi B, Tayebi A. A new quantity in Finsler geometry. C R Math 2011;349:81–83. [CrossRef]
[18] Bao D, Chern SS. On a notable connection in Finsler Geometry. Houston J Math 1993;19:135–180.
[19] Tayebi A, Azizpour E, Esrafilian E. On a Family of connections in Finsler geometry. Publ Math Debrecen 2008;72:1–15. [CrossRef]
[20] Ingarden RS. On the geometrically absolute optical representation in the electron microscope. Trav Soc Sci Lettr Wroclaw. 1957;B45:3–60.
[21] Bao D, Chern SS, Shen Z. An Introduction to Riemann-Finsler Geometry. New York: Springer; 2000. [CrossRef]
[22] Cheng X, Mo X, Shen Z. On the flag curvature of Finsler metrics of scalar curvature. J London Math Soc 2003;68:762–780. [CrossRef]
[23] Cheng X, Shen Z. A class of Finsler metrics with isotropic S-curvature. Isr J Math 2009;169:317–340. [CrossRef]
[24] Cheng X, Shen Z. Randers metrics of scalar flag curvature. J Australian Math Soc 2009;87:359–370. [CrossRef]
[25] Chern SS, Shen Z. Riemann-Finsler Geometry. Singapore: World Scientific; 2005. [CrossRef]
[26] Tamassy L. Metrical almost linear connections in TM for Randers spaces. Bull Soc Sci Lett Lodz Ser Rech Deform 2006;51:147–152.
[27] Shen Z. Lectures on Finsler Geometry. Singapore: World Scientific Publishers;2001. [CrossRef]