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Research Article

New Analytical Solutions of Heisenberg Ferromagnetic Spin Chain with Functional Variable Method

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Abstract

The Heisenberg spin chain concept is a fundamental and generic model that describes the exotic magnetic behavior of certain materials, such as ferromagnetism, antiferromagnetism, and ferrimagnetism under critical temperatures. The concept of spin chain is based on Coulomb interactions due to Pauli exclusion principle rather than dipole-dipole interactions in explaining the high energy observed in the Weiss molecular field. With certain improvements to the Hamiltonian proposed by Heisenberg, the model has became more sophisticated and used successfully in explaining many of the physical phenomena observed experimentally. This model has been extensively studied by physicists since the emergence of quantum physics at the beginning of the 20th century. Due to nonlinear interactions inherent in the model, soliton solutions that can be obtained have attracted the attention of mathematicians, in recent decades. In this study, triangular soliton, bell shaped solitary wave and kink shaped solitary wave solutions were obtained by applying the functional variable method to the nonlinear Heisenberg spin chain equation for a cubic lattice crystal.

Keywords: Heisenberg Ferromagnetic Spin Chain Equation, Analytical Solution, Functional Variable Method.

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1. Introduction

Phase transitions are very important for any material [1]. In most cases materials properties (such as electrical, mechanical, thermodynamic and magnetic etc.) dramatically changes with the phase transition. For instance, the magnetization of a ferromagnetic material increases with increasing temperatures until a critical temperature so-called the Curie temperature and turns into a paramagnetic material with further increase. Or, a similar condition is valid for an antiferromagnetic material above a critical temperature so-called the Néel temperature. In the early 1900 the origin of exotic phenomena of ferromagnetism, ferrimagnetism and antiferromagnetism tried to be explained by many scientists. Weiss comes up with the idea of magnetic domains which does not cause magnetism to a material when they randomly oriented but cause to magnetism with highly ordering by an external magnetic field. It is very complicated and nearly impossible to cover all the interaction of the magnetic moments of every single molecule of a crystal. It is inevitable to make some approaches based on the phenomena in the microstructure in explaining the phenomena seen in the macrostructure. Weiss introduced the molecular field model which assumes mean field is proportional to magnetization [2]. This model was successful in explaining specific heat anomaly and spontaneous magnetization. However, it failed to explain the origin of the molecular field that is proposed. The source of this field remained a mystery until the epoch of quantum physics came into play. This molecular field had to be able to produce energy that could not be explained by classical dipole-dipole interactions. Heisenberg realized that this field cannot be explained without quantum exchange interactions arising from the Pauli exclusion principle requiring the antisymmetric wave function with an exchange of space and spin coordinates. In other words, the source of this field had to be Coulomb interactions instead of magnetic interactions that have very few energies. In other words, the source of this field had to be Coulomb interactions instead of magnetic interactions that have very few energies.

By considering only the interatomic interactions between the atoms at the sites of a lattice and the pauli exclusion principle (the requirement that the total wave function should be asymmetric), Heisenberg put forward the following Hamiltonian [3];

$$H = -\sum_{i,j} \zeta_{ij} \overleftarrow{s_i} \cdot \overleftarrow{s_j} \tag{1.1}$$

with

$$\zeta_{ij} = \begin{cases} J, & \text{if } \ i,j \ \text{ are neigbors} \\ 0, & \text{else} \end{cases} \tag{1.2}$$

where ζ_{ij} is coupling constant of two neighbor atoms, $\overleftarrow{s_i}$ and $\overleftarrow{s_j}$ are Spin vectors at i-th and j-th site, respectively. The spin chain concept is also due to these interatomic interactions. Heisenberg chain model is a very generic and essential model for magnetism in matter. Other interactions were added to the proposed Hamilton model and the sophistication of the model was studied extensively [4-7].

The (2+1)-dimensional Heisenberg ferromagnetic spin chain equation is given by [8-10]

$$i\psi_t + \alpha_1\psi_{xx} + \alpha_2\psi_{yy} + \alpha_3\psi_{xy} - \alpha_4|\psi|^2\psi = 0, \quad i = \sqrt{-1} \tag{1.3}$$

where $\alpha_1 = \gamma^4(\beta + \beta_2), \alpha_2 = \gamma^4(\beta_1 + \beta_2), \alpha_3 = 2\gamma^4\beta_2$ and $\alpha_4 = 2\gamma^4 A$. And, γ represents the lattice parameter that is related to the crystal structure of a solid material. It differs for each material and each unit cell, β, β_1 represent the coefficients of bilinear exchange interactions along the x- and y-directions, respectively. On the other hand, diagonal neighboring interaction coefficient and uniaxial crystal field anisotropy parameter are given by β_2 and A, respectively.

In this article functional variable method is considered to get the exact solutions of (2+1)-dimensional Heisenberg ferromagnetic spin chain equation. To the best of our knowledge all the solutions are new and never seen in the literature.

The soliton solutions derived in this work have numerous practical applications. For instance, they can be tested experimentally using magnetic resonance techniques or neutron scattering to observe soliton behavior and domain wall motion in materials like yttrium iron garnet or perovskite manganites. The bell-shaped solitons could correspond to spin-wave excitations that are useful in designing magnon-based logic devices. Furthermore, controlling domain walls through kink solitons offers potential in spintronic devices, where magnetic states are manipulated for data storage or non-volatile memory. Future experimental setups could use these solutions to explore high-temperature ferromagnetism or quantum phase transitions, providing deeper insights into material behavior under extreme conditions.

In this work, we have derived various soliton solutions, including triangular, bell-shaped, and kink solitons. These solutions have significant implications for understanding domain wall dynamics and phase transitions in ferromagnetic materials. For example, the triangular soliton may correspond to sharp transitions between magnetic domains under external fields. The bell-shaped solitons describe localized spin wave excitations, which could play a role in the thermal stability of the magnetization. Kink solitons, on the other hand, represent phase transitions between different magnetic states, potentially providing insights into the movement of domain walls in the presence of anisotropy or external magnetic fields. These soliton solutions offer a mathematical framework that could be experimentally verified in ferromagnetic materials through methods such as neutron scattering or magnetic resonance imaging, offering new ways to control or manipulate magnetic states in applied technologies such as spintronics or magnetic memory.

2. Description of Functional Variable Method

Functional variable method provides more accurate traveling wave solutions with additional free parameters with respect to other methods. This a great advantage over the other proposed methods. Functional variable method can simply be explained as follows [11-13]. Let us take into account a nonlinear partial differential equation in the form of;

$$P\left(\psi, \frac{\partial \psi}{\partial t}, \frac{\partial \psi}{\partial x}, \frac{\partial \psi}{\partial y}, \frac{\partial^2 \psi}{\partial t^2}, \frac{\partial^2 \psi}{\partial x^2}, \frac{\partial^2 \psi}{\partial y^2}, \dots\right) = 0. \tag{2.1}$$

where ψ is a space and time dependent arbitrary function and can be determined as

$$\psi(x,y,t) = \Phi(\xi), \xi = ax + by - vt. \tag{2.2}$$

Here, a, b and v are coefficients of the x, y and t variables, respectively. With the aid of the transform given by Eq.(2.2), the Eq.(2.1) becomes into an ordinary differential equation Eq.(2.3).

$$Q(\Phi, \Phi_\xi, \Phi_{\xi\xi}, \Phi_{\xi\xi\xi}, \dots) = 0. \tag{2.3}$$

In order to obtain an unknown function Φ, let us take into account a functional variable such as

$$\Phi_\xi = F(\Phi). \tag{2.4}$$

Some derivatives of Φ given in Eq.(2.5) can be obtained from the Eq.(2.4).

$$\Phi_{\xi\xi} = \frac{1}{2}(F^2)',$$

$$\Phi_{\xi\xi\xi} = \frac{1}{2}(F^2)''\sqrt{F^2},$$

$$\Phi_{\xi\xi\xi\xi} = \frac{1}{2}[(F^2)'''F^2 + (F^2)''(F^2)'],$$

$$\vdots \tag{2.5}$$

where "$'$" indicates the $\frac{d}{d\Phi}$. An ordinary differential equation given in Eq.(2.6) which depends on Φ, F and the derivatives of F upon Φ can be obtained by putting the (2.5) into (2.3).

$$R(\Phi, F, F', F'', F''', F^{(4)}, \dots) = 0. \tag{2.6}$$

The solutions of Eq.(2.1) can be acquired by using F (obtained by integrating Eq.(2.6)) in Eq.(2.4).

3. Analytical Solutions of Heisenberg Ferromagnetic Spin Chain Equation

Let us take the following complex wave transformation:

$$\psi(x,y,t) = \Phi(\xi)e^{i\Omega}, \xi = ax + by - vt, \Omega = px + qy - rt. \tag{3.1}$$

Substituting Eq.(3.1) into Eq.(1.3), gives the following relationship

$$(r - \alpha_1 p^2 - q(\alpha_2 q + \alpha_3 p))\Phi - \alpha_4\Phi^3 + (\alpha_1 a^2 + \alpha_2 b^2 + \alpha_3 ab)\Phi'' = 0 \tag{3.2}$$

and

$$v = 2a\alpha_1 p + 2b\alpha_2 q + \alpha_3(bp + aq).$$

By substituting $\Phi'' = \frac{1}{2}(F^2)'$ into (3.2) gives the following equation

$$F(\Phi) = \Phi' = \sqrt{-\frac{r-\alpha_1 p^2 - q(\alpha_2 q + \alpha_3 p)}{\alpha_1 a^2 + \alpha_2 b^2 + \alpha_3 ab}\Phi^2 + \frac{\alpha_4}{2(\alpha_1 a^2 + \alpha_2 b^2 + \alpha_3 ab)}\Phi^4 + h} \tag{3.3}$$

where h is an integration constant. We get solutions of Eq. (1.3) by solving Eq. (3.3) as follows:

Solution 1 For $h = 0$, $\frac{r-\alpha_1 p^2 - q(\alpha_2 q + \alpha_3 p)}{\alpha_1 a^2 + \alpha_2 b^2 + \alpha_3 ab} > 0$ and $\frac{\alpha_4}{2(\alpha_1 a^2 + \alpha_2 b^2 + \alpha_3 ab)} > 0$, Eq. (1.3) has triangular soliton solutions as following:

$$\psi_{1,2}(x,y,t) = \pm\sqrt{\frac{2(-\alpha_1 p^2 - q(\alpha_3 p + \alpha_2 q) + r)}{\alpha_4}}\sec\left(\sqrt{\frac{-\alpha_1 p^2 - q(\alpha_3 p + \alpha_2 q) + r}{a^2\alpha_1 + a\alpha_3 b + \alpha_2 b^2}}\xi\right)e^{i\Omega},$$

$$\psi_{3,4}(x,y,t) = \pm\sqrt{\frac{2(-\alpha_1 p^2 - q(\alpha_3 p + \alpha_2 q) + r)}{\alpha_4}}\csc\left(\sqrt{\frac{-\alpha_1 p^2 - q(\alpha_3 p + \alpha_2 q) + r}{a^2\alpha_1 + a\alpha_3 b + \alpha_2 b^2}}\xi\right)e^{i\Omega}.$$

Solution 2 For $h = 0$, $\frac{r-\alpha_1 p^2 - q(\alpha_2 q + \alpha_3 p)}{\alpha_1 a^2 + \alpha_2 b^2 + \alpha_3 ab} < 0$ and $\frac{\alpha_4}{2(\alpha_1 a^2 + \alpha_2 b^2 + \alpha_3 ab)} < 0$, we get bell shaped solitary wave solutions for Eq. (1.3) as following:

$$\psi_{5,6}(x,y,t)=\pm\sqrt{\frac{2(-\alpha_1p^2-q(\alpha_3p+\alpha_2q)+r)}{\alpha_4}}\operatorname{sech}\left(\sqrt{\frac{\alpha_1p^2+q(\alpha_3p+\alpha_2q)-r}{a^2\alpha_1+a\alpha_3b+\alpha_2b^2}}\,\xi\right)e^{i\Omega},$$

$$\psi_{7,8}(x,y,t)=\pm i\sqrt{\frac{2(-\alpha_1p^2-q(\alpha_3p+\alpha_2q)+r)}{\alpha_4}}\operatorname{csch}\left(\sqrt{\frac{\alpha_1p^2+q(\alpha_3p+\alpha_2q)-r}{a^2\alpha_1+a\alpha_3b+\alpha_2b^2}}\,\xi\right)e^{i\Omega}.$$

Solution 3 For $h=\frac{(-\alpha_1p^2-q(\alpha_3p+\alpha_2q)+r)^2}{2\alpha_4(a^2\alpha_1+a\alpha_3b+\alpha_2b^2)}$, $\frac{r-\alpha_1p^2-q(\alpha_2q+\alpha_3p)}{\alpha_1a^2+\alpha_2b^2+\alpha_3ab}>0$ and $\frac{\alpha_4}{2(\alpha_1a^2+\alpha_2b^2+\alpha_3ab)}>0$, we get kink shaped solitary wave solutions for Eq. (1.3) as following:

$$\psi_{9,10}(x,y,t)=\pm\sqrt{\frac{-\alpha_1p^2-q(\alpha_3p+\alpha_2q)+r}{\alpha_4}}\tanh\left(\sqrt{\frac{-\alpha_1p^2-q(\alpha_3p+\alpha_2q)+r}{2(a^2\alpha_1+a\alpha_3b+\alpha_2b^2)}}\,\xi\right)e^{i\Omega},$$

$$\psi_{11,12}(x,y,t)=\pm\sqrt{\frac{-\alpha_1p^2-q(\alpha_3p+\alpha_2q)+r}{\alpha_4}}\coth\left(\sqrt{\frac{-\alpha_1p^2-q(\alpha_3p+\alpha_2q)+r}{2(a^2\alpha_1+a\alpha_3b+\alpha_2b^2)}}\,\xi\right)e^{i\Omega}.$$

Solution 4 For $h=\frac{(-\alpha_1p^2-q(\alpha_3p+\alpha_2q)+r)^2}{2\alpha_4(a^2\alpha_1+a\alpha_3b+\alpha_2b^2)}$, $\frac{r-\alpha_1p^2-q(\alpha_2q+\alpha_3p)}{\alpha_1a^2+\alpha_2b^2+\alpha_3ab}<0$ and $\frac{\alpha_4}{2(\alpha_1a^2+\alpha_2b^2+\alpha_3ab)}>0$, for Eq. (1.3), we obtain triangular soliton solutions as following:

$$\psi_{13,14}(x,y,t)=\pm\sqrt{\frac{\alpha_1p^2+q(\alpha_3p+\alpha_2q)-r}{\alpha_4}}\tan\left(\sqrt{\frac{\alpha_1p^2+q(\alpha_3p+\alpha_2q)-r}{2(a^2\alpha_1+a\alpha_3b+\alpha_2b^2)}}\,\xi\right)e^{i\Omega},$$

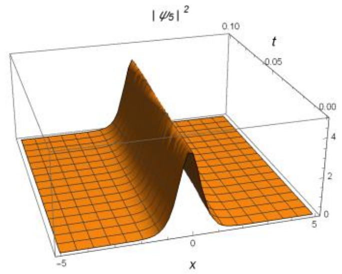
$$\psi_{15,16}(x,y,t)=\pm\sqrt{\frac{(\alpha_1p^2+q(\alpha_3p+\alpha_2q)-r)}{\alpha_4}}\cot\left(\sqrt{\frac{\alpha_1p^2+q(\alpha_3p+\alpha_2q)-r}{2(a^2\alpha_1+a\alpha_3b+\alpha_2b^2)}}\,\xi\right)e^{i\Omega}.$$

4. Applications of Some Solutions

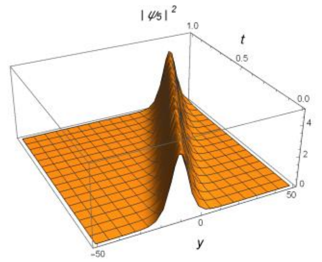
In this section, solutions of $\psi_5(x,y,t)$ and $\psi_9(x,y,t)$ are graphically represented to see the spatial-temporal distribution of probabilities of the wave function. In figure 1, a bell shaped solitary wave form can be seen for the probability along x and y directions with time. On the other hand, a double kink shaped solitary wave forms along x axis, a single kink shaped solitary wave forms along y axis can be seen for the probability with time in figure 2(a) and 2(b), respectively. And consequently, a double kink shaped solitary wave form can be seen for the spatial distribution of probability.

Probability distribution of the wave function $\psi_5(x,y,t)$ showing a bell-shaped solitary wave. The figure illustrates the localized nature of the soliton along both the x and y directions over time, indicating stable spin wave excitations. These solutions are relevant for experimental studies of soliton interactions in low-temperature ferromagnetic materials.

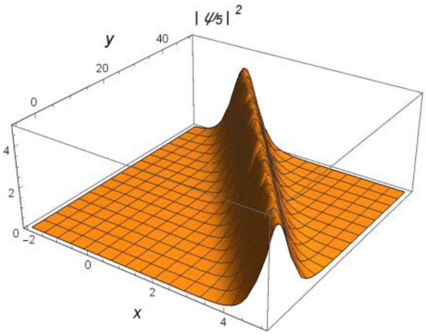
Probability distribution of the wave function $\psi_9(x,y,t)$, showing kink-shaped solitary waves. The kink solitons illustrate phase transitions between magnetic domains. The double-kink shape along the x-axis and single-kink along the y-axis demonstrate the anisotropic nature of the soliton behavior, potentially corresponding to domain wall motion in ferromagnetic materials.



(a)$y = 1$

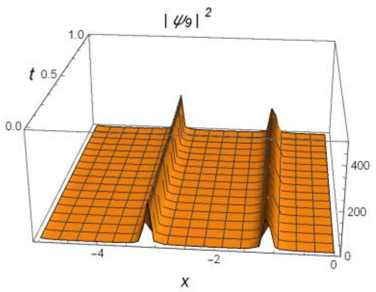


(b)$x = 1$

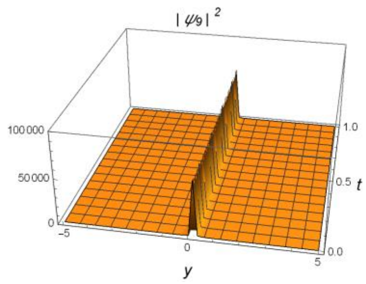


(c)$t = 1$

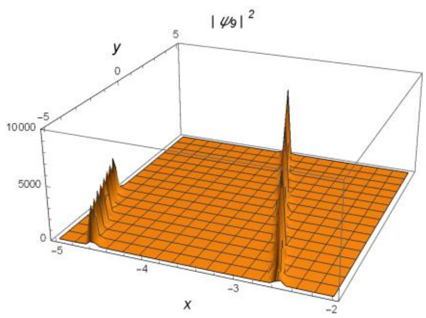
Figure 1: Probability distribution of the wave function $\psi_5(x, y, t)$ that illustrates a bell-shaped solitary wave for the parametres of $r = 1, \gamma = 1, \beta = 0.1, \beta_2 = 1, \beta_1 = 1, A = -1, p = 1, q = 1, a = 1, b = 0.1$.



(a)$y = 1$



(b)$x = 1$



(c)$t = 1$

Figure 2: Probability distribution of the wave function $\psi_9(x, y, t)$ that corresponds to kink-shaped solitary waves.for the parametres of $r = 1,\ \gamma = 4,\ \beta = 0.1,\ \beta_2 = 0.1,\ \beta_1 = 1,\ A = 1,\ p = -1,\ q = 1,\ a = 1,\ b = 0.1$.

5. Conclusion

The Heisenberg chain model equation which is arising in describing ferromagnetism, antiferromagnetism and ferrimagnetism phenomena seen below a critical temperature of certain materials is solved by functional variable method with the help of computer software called Wolfram Mathematica. Sixteen solutions which have form of triangular soliton, bell shaped solitary wave and kink shaped solitary wave are obtained for a cubic lattice crystal. Some of the solutions are graphically represented to see the spatial and temporal variations and probability distribution of the wave function. One can easily obtain the expected values of energy, momentum or any other observables by applying the quantum operators to the wave functions attained in this study.

Abbreviations

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Authors' contributions

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Competing interests

The authors declare that they have no competing interests.

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